ON COMPACT RIEMANNIAN MANIFOLDS WITH NONCOMPACT HOLONOMY GROUPS

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Abstract

Solving a long standing problem in Riemannian geometry we construct a compact Riemannian manifold with a noncompact holonomy group. As the title indicates we then prove structure theorems for these manifolds. We employ an argument of Cheeger and Gromoll [1971] to show that the holonomy group of a compact Riemannian manifold is compact if and only if the image of the so called holonomy representation of its fundamental group is finite. Then we characterize these holonomy representations algebraically. As a consequence we prove that a finite cover of a compact Riemannian manifold $M^{(n)}$ with a noncompact holonomy group is the total space of a torus bundle over another compact Riemannian manifold $B^{(b)}$ with $b \leq n - 4$.

1. Introduction

For a Riemannian manifold M and a piecewise smooth curve $\gamma \colon [0,1] \to M$ we let $\mathrm{Par}_\gamma \colon T_{\gamma(0)}M \to T_{\gamma(1)}M$ denote the parallel transport along γ. The holonomy group $\mathrm{Hol}(M, p)$ of M at p is the subgroup of the orthogonal group $\mathsf{O}(T_p M)$ consisting of all endomorphisms of the form Par_γ where γ runs over all loops at p.

It is an elementary consequence of the de Rahm decomposition theorem [2, Theorem 10.43] that the holonomy group of a simply connected, complete Riemannian manifold is compact. For the general case the best result so far known is due to Cheeger and Gromoll [4, Theorem 6]. They combined their splitting theorem with the holonomy classification theorem of Berger [1] and Simons [9] to show that the holonomy group of a compact Riemannian manifold is compact, provided that the Euclidean factor of the universal covering space is either at most one-dimensional

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or cocompact. In fact, the assumptions of this theorem turn out to be optimal:

Example 1.   There is a five-dimensional, simply connected, solvable Lie group S, a left invariant metric g on S and a discrete, cocompact subgroup $\Lambda \subset S$ such that the holonomy group of the compact Riemannian manifold $M := \Lambda \backslash (S, g)$ is isomorphic to $SO(3) \times \mathbb{Z}$.

After learning that compact Riemannian manifolds with noncompact holonomy groups exist, we want to investigate the structure of these manifolds. More generally, we will prove structure theorems for a compact manifold M whose universal covering space has a nontrivial Euclidean factor. For that it is crucial to define and to understand the holonomy representation of a fundamental group of a compact Riemannian manifold: For a connected compact Riemannian manifold M and a point $p \in M$ we consider the subspace V of $T_p M$ that is kept pointwise fixed by all elements of the identity component $\mathrm{Hol}_0(M, p)$ of $\mathrm{Hol}(M, p)$. It is easy to see that for a loop γ at p the restriction of Par_γ to V only depends on the homotopy class $[\gamma] \in \pi_1(M, p) =: \pi_1(M)$. The holonomy representation $\rho_1 \colon \pi_1(M) \to O(V)$ is defined by $\rho_1([\gamma]) := \mathrm{Par}_{\gamma|V}$. Finally, for any representation $\rho \colon \Pi \to GL(W)$ in a real vector space W we define the integer rank of ρ by

$$(1) \qquad \mathbb{Z}\text{-rk}(\rho) := \min \left\{ \mathrm{rank}(\Lambda) \;\middle|\; \begin{array}{c} \Lambda \text{ is a finitely generated,} \\ \rho\text{-invariant, cocompact} \\ \text{subgroup of } W \end{array} \right\}.$$

We emphasize that the group Λ occurring in this definition is not necessarily discrete. Moreover, $\mathbb{Z}\text{-rk}(\rho) = \infty$ is allowed.

Theorem 2.   *Let (M, g) be a connected, compact Riemannian manifold, $p \in M$, and let $\rho_1 \colon \pi_1(M) \to O(V)$ be the natural holonomy representation as introduced above. Then the fundamental group $\pi_1(M)$ contains a finitely generated, free abelian normal subgroup L for which the following hold:*

(i) *The natural action of $\pi_1(M)$ on L by conjugation induces a representation $\pi_1(M) \to GL(L)$ in the $\mathbb{Z}$-module L and a corresponding real representation $\tilde{\rho} \colon \pi_1(M) \to GL(L \otimes_{\mathbb{Z}} \mathbb{R})$ in $L \otimes_{\mathbb{Z}} \mathbb{R}$.*

(ii) *The representation $\tilde{\rho}$ decomposes as a direct sum $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$ where $\tilde{\rho}_1$ is equivalent to ρ_1.*

(iii) There is a subgroup $\mathsf{H} \subset \pi_1(M)$ satisfying $\mathsf{L} \cap \mathsf{H} = \{e\}$, and $\mathsf{H} \cdot \mathsf{L}$ is of finite index in $\pi_1(M)$.

(iv) $\mathrm{rank}(\mathsf{L}) = \mathbb{Z}\text{-}\mathrm{rk}(\rho_1) < \infty$.

In the special case of a compact flat manifold M the theorem is well-known: It is an elementary consequence of the first Bieberbach theorem that the holonomy representation of $\pi_1(M)$ can be obtained by letting $\pi_1(M)$ operate by conjugation on its translational part.

Notice that statement (iv) yields immediately the inequality $\mathrm{rank}(\mathsf{L}) \geq \dim(V)$. Because of the de Rahm decomposition theorem the dimension of V is equal to the dimension of the Euclidean factor of the universal covering space of M. Thus this inequality can be viewed as a partial generalization of the first Bieberbach theorem.

As we will see in Section 3, the arguments in the proof of the above quoted theorem of Cheeger and Gromoll can be used to prove that the holonomy group of M is compact if and only if the image of the holonomy representation $\rho_1 \colon \pi_1(M) \to \mathsf{O}(V)$ is finite. This allows us to show:

Corollary 3. *Let M be a connected, compact Riemannian manifold with a noncompact holonomy group. Then there is a nontrivial, finitely generated, free abelian normal subgroup $\mathsf{L} \subset \pi_1(M)$ and an element $g \in \pi_1(M)$ such that the $\mathbb{Z}$-linear map $c_g \colon \mathsf{L} \to \mathsf{L}$, $v \mapsto gvg^{-1}$ has an eigenvalue $\lambda \in \mathsf{S}^1 \subset \mathbb{C}$ which is not a root of unity. Moreover, $\mathrm{rank}(\mathsf{L}) \geq \dim(V) + 2 \geq 4$, where V is defined as in Theorem 2.*

Furthermore, in combination with its proof Theorem 2 yields the following corollary, which can be viewed as the main structure result of this paper.

Corollary 4. *Let (M, g), p, $V \subset T_p M$, ρ_1 be as in Theorem 2. Then for a finite Riemannian cover $(\hat{M}, g)$ of (M, g) the following is true: There is a compact Riemannian manifold B and a Riemannian submersion $\sigma \colon \hat{M} \to B$ such that the fibers of σ are flat tori of dimension $d = \mathbb{Z}\text{-}\mathrm{rk}(\rho_1) \geq \dim(V)$. Moreover, there is a smooth section $s \colon B \to \hat{M}$. Finally, we have the following in addition:*

a) If the holonomy group of M is compact, then the fibers of σ are the orbits of a free isometric torus action, and the corresponding differentiable principal torus bundle over B is trivial. In particular, $\hat{M}$ is diffeomorphic to the product $B \times (\mathsf{S}^1)^d$.

b) If the holonomy group of M is noncompact, then

$$d \geq \dim(V) + 2 \geq 4.$$

c) If the Ricci curvature of M is nonpositive, then the horizontal distribution of σ is integrable.

d) If the sectional curvature of M is nonpositive, then fibers of σ are totally geodesic. Moreover, the holonomy group of M is compact.

e) If the Ricci tensor of M is parallel, then $\mathrm{Hol}(M, p)$ is compact.

Corollary 4 yields obstructions for a manifold to have a prescribed holonomy representation. For example, let M be a compact manifold with a compact holonomy group, and suppose that the image of the holonomy representation $\rho_1 \colon \pi_1(M) \to \mathsf{O}(V)$ contains an element of order k. Then $\mathbb{Z}\text{-rk}(\rho_1) \geq \varphi(k)$, where φ denotes Euler's φ-function. Thus, by part a) of the addendum, a finite cover $\hat{M}$ of M is diffeomorphic to $B \times (\mathsf{S}^1)^{\varphi(k)}$ for some compact manifold B. This is to some extent a striking conclusion, since the number $\varphi(k)$ can be huge even if $\dim(V) = 2$.

In the light of Corollary 4 b) it is not very surprising that there is much more rigidity in low dimensions:

Corollary 5.   *Let M be a connected, compact Riemannian manifold with a noncompact holonomy group.*

a) Then $\dim(M) \geq 5$ and in the case $\dim(M) = 5$ the following hold: There is a connected, simply connected, five-dimensional, solvable Lie group S, a finite group $\mathsf{F} \subset \mathrm{Aut}(\mathsf{S})$ and a discrete, cocompact, torsion free subgroup $\Gamma \subset \mathsf{S} \rtimes \mathsf{F}$ such that M is diffeomorphic to the quotient S/Γ. Moreover, M also admits a local homogeneous metric with a noncompact holonomy group.

b) If $\dim(M) = 6$, then M is aspherical.

From part a) of the corollary it becomes apparent that the manifold of Example 1 is a solvmanifold not just by accident.

For the next theorem we introduce a notation: We say that a real representation $\psi \colon \Pi \to \mathsf{GL}(U)$ of a finitely generated group Π is a holonomy representation if and only if the following hold: There is a compact Riemannian manifold M and an isomorphism $\iota \colon \Pi \to \pi_1(M)$ such that $\rho_1 \circ \iota \colon \Pi \to \mathsf{O}(V)$ is equivalent to ψ, where $\rho_1 \colon \pi_1(M) \to \mathsf{O}(V)$ is the

holonomy representation of $\pi_1(M)$ as introduced above. We can now formulate a converse statement to Theorem 2.

Theorem 6. *Let Π be a finitely generated and finitely presented group. Suppose there is a finitely generated, free abelian normal subgroup $\mathsf{L} \subset \Pi$ and a subgroup $\mathsf{H} \subset \Pi$ for which the following two conditions are satisfied.*

(i) $\mathsf{H} \cap \mathsf{L} = \{e\}$ and $\mathsf{H} \cdot \mathsf{L}$ is of finite index in Π.

(ii) The representation $\tilde{\rho}\colon \Pi \to \mathsf{GL}(\mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R})$ that is induced by conjugation decomposes, $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$, such that $\tilde{\rho}_1$ is equivalent to an orthogonal representation.

Then $\tilde{\rho}_1$ is a holonomy representation.

Notice that in combination the two theorems characterize the holonomy representations of the fundamental groups of compact Riemannian manifolds. As is shown in the proof of Theorem 6 the representation $\tilde{\rho}_1$ can be realized as the holonomy representation of the fundamental group of a compact n-manifold with $n = \mathbb{Z}\text{-rk}(\tilde{\rho}_1) + 4$. On the other hand, by Corollary 4 the integer rank $\mathbb{Z}\text{-rk}(\rho_1)$ of the holonomy representation of the fundamental group of any compact Riemannian manifold M is bounded by the dimension of M. So we have a quite good estimate for the minimal dimension in which a given holonomy representation occurs.

Using Theorem 6 it is easy to characterize the images of holonomy representations as well:

Corollary 7. *A finitely generated subgroup Φ of the orthogonal group $\mathsf{O}(l)$ is the image of a holonomy representation if and only if there is a finitely generated, cocompact subgroup $\Lambda \subset \mathbb{R}^l$ that is invariant under Φ.*

Finally, we show that there are no obstructions for the closures of the images of holonomy representations:

Corollary 8. *Let G be a closed subgroup of the orthogonal group $\mathsf{O}(l)$. Then there is a finitely generated dense subgroup $\Phi \subset \mathsf{G}$ which is the image of a holonomy representation.*

Of course, Corollary 8 ensures the existence of many compact Riemannian manifolds with rather exotic holonomy representations.

Remarks.

1. In subsection 2.1 we construct a compact Kähler manifold (N, g) with a noncompact holonomy group and $\dim_{\mathbb{C}}(N) = 5$. We do not know whether five is the minimal complex dimension in which compact Kähler manifolds with noncompact holonomy groups occur, nor do have an analogue of Theorem 6 for the holonomy representations of fundamental groups of compact Kähler manifolds. So it might be true that there are additional obstructions for these representations.

2. Observe that Corollary 4 e) applies to locally symmetric spaces, Einstein manifolds and quaternion Kähler manifolds. Thus the compact manifolds in these classes have compact holonomy groups.

3. Viewing at Corollary 4 c) one might ask whether there exists a compact Riemannian manifold with nonpositive Ricci curvature and a noncompact holonomy group. The answer is yes. In fact, a straightforward computation shows that the manifold of Example 1 can be chosen such that its Ricci curvature is nonpositive.

4. It is clear from the proof in subsection 6.1 that the conclusion of Corollary 4 d) remains valid if the hypothesis "K $\leq$ 0" is replaced by the following weaker assumption: For all $q \in M$ and all orthonormal vectors $v_1, \ldots, v_d \in T_q M$ the following two inequalities hold

$$\sum_{i=1}^{d} \mathrm{Ric}(v_i, v_i) \leq 0 \quad \text{and} \quad \sum_{\substack{i,j=1 \\ i \neq j}}^{d} \mathrm{K}\big(\mathrm{span}_{\mathbb{R}}(v_i, v_j)\big) \leq 0.$$

5. Combining Corollary 5 a) with a theorem of [5], we conclude that two five-dimensional compact Riemannian manifolds with noncompact holonomy groups are diffeomorphic if and only if their fundamental groups are isomorphic. For further structure theorems on infrasolvmanifolds we refer to [12].

Up to some extensions and improvements the paper is also part of the author's dissertation, [11]. I am grateful to my Ph.D. advisor Professor W.T. Meyer for his enlightening lectures on Riemannian geometry.

2. A compact 5-manifold with a noncompact holonomy group

Let $\varphi \in (0, 2\pi)$ be the number with $e^{i\varphi} = 1 - \sqrt{2} + i\left(\sqrt{2} \cdot \sqrt{\sqrt{2} - 1}\right)$, and set $\lambda := 1 + \sqrt{2} \cdot \left(1 + \sqrt{\sqrt{2} + 1}\right)$. It is straightforward to check that the two matrices

(2)
$$A = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

have the same characteristic polynomial. Since both matrices are semisimple, there is a matrix $S \in \mathsf{GL}(n, \mathbb{R})$ satisfying $A = SBS^{-1}$. Notice that B is contained in $\mathsf{GL}(4, \mathbb{Z})$, and hence the cyclic group generated by A leaves the lattice $\mathsf{L} := S \cdot \mathbb{Z}^4$ invariant. It is easy to see that there is a matrix $X \in M(4, \mathbb{R})$ with $\exp(X) = A$. Moreover, we can assume that X has the same block form as A and that the first block $(x_{ij})_{i,j=1,2}$ is the skew symmetric matrix $\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}$. Consider the semidirect product

$$\mathsf{S} := \mathbb{R}^4 \rtimes \mathbb{R}, \quad (v, s) \cdot (w, t) := \left(v + \exp(sX)w, s + t\right)$$

and the discrete, cocompact subgroup $\Lambda := \mathsf{L} \rtimes \mathbb{Z}$. We view the two subspaces $\mathsf{U} = \mathbb{R}^2 \times \{0\}$ and $\mathsf{V} = \{0\} \times \mathbb{R}^2$ in the natural fashion as normal subgroups of S. The group $\mathsf{S}_1 := \mathsf{V} \rtimes \mathbb{R} \subset \mathsf{S}$ is complementary to U, that is $\mathsf{U} \cap \mathsf{S}_1 = \{e\}$ and $\mathsf{S} = \mathsf{S}_1 \cdot \mathsf{U}$. Thus we can describe S also as a semidirect product $\mathsf{U} \rtimes_\beta \mathsf{S}_1$, where $\beta \colon \mathsf{S}_1 \to \mathsf{GL}(\mathsf{U})$ is the representation induced by conjugation.

Observe that β is an orthogonal representation with respect to the natural scalar product $\langle \cdot, \cdot \rangle$ on $\mathsf{U} \subset \mathbb{R}^4$. Choose on S_1 a left invariant metric g_1. Since β is an orthogonal representation, the product metric $g = \langle \cdot, \cdot \rangle \times g_1$ is a left invariant metric on $\mathsf{U} \rtimes_\beta \mathsf{S}_1 \cong \mathsf{S}$. The manifold (S_1, g_1) does not split isometrically as a product, and therefore the holonomy group of (S, g) is isomorphic to $\mathsf{SO}(3)$.

Let $\mathsf{L}_h \colon \mathsf{S} \to \mathsf{S}$, $a \mapsto h \cdot a$ be the left translation for $h \in \Lambda \subset \mathsf{S}$. For a curve $\gamma \colon [0, 1] \to \mathsf{S}$ from the neutral element e to $h \in \Lambda$ we define

$\overline{\mathrm{Par}}_\gamma := \mathrm{L}_{h^{-1}*} \circ \mathrm{Par}_\gamma \colon \mathfrak{s} \to \mathfrak{s}$, where $\mathrm{Par}_\gamma \colon \mathfrak{s} = T_e\mathsf{S} \to T_h\mathsf{S}$ denotes the parallel transport along γ. Evidently, the holonomy group of the quotient $\Lambda\backslash(\mathsf{S}, g)$ is isomorphic to the subgroup Φ of $\mathsf{O}(\mathfrak{s})$ consisting of all endomorphisms of the form $\overline{\mathrm{Par}}_\gamma$ where γ runs over all curves connecting e with some element $h \in \Lambda \subset \mathsf{S}$.

The group Φ leaves the subalgebras $\mathfrak{s}_1$ and $\mathfrak{u}$ corresponding to the subgroups S_1 and U invariant. Furthermore, the image of the natural projection $\Phi \to \mathsf{O}(\mathfrak{s}_1)$ is $\mathsf{SO}(\mathfrak{s}_1) \cong \mathsf{SO}(3)$. It only remains to check that the image of the projection $\mathrm{pr}\colon \Phi \to \mathsf{O}(\mathfrak{u})$ is an infinite cyclic group. Notice that for a curve γ from e to h the endomorphism $\overline{\mathrm{Par}}_{\gamma|\mathfrak{u}}$ only depends on h and that $\overline{\mathrm{Par}}_{\gamma|\mathfrak{u}} = \mathrm{Ad}_{h^{-1}|\mathfrak{u}}$. Consequently, the image of pr is the group $\mathsf{A} := \{\mathrm{Ad}_{h|\mathfrak{u}} \mid h \in \Lambda\}$.

Clearly, $\mathrm{Ad}_{h|\mathfrak{u}} = \mathrm{id}$ for $h \in \mathsf{L} \subset \mathsf{L} \rtimes \mathbb{Z} = \Lambda$. Moreover, for $h_0 = (0, 1) \in \mathsf{L} \rtimes \mathbb{Z}$ the map $\mathrm{Ad}_{h_0|\mathfrak{u}}$ is a rotation by the angel φ. Since $e^{i\varphi}$ is not a root of unity, it follows that A is an infinite cyclic group.

2.1   A compact Kähler manifold with a noncompact holonomy group

We consider again the numbers $\lambda, \varphi \in \mathbb{R}$ that we have used in equation (2) to define the matrix A. Notice that the map

$\alpha\colon \mathbb{C} \to \mathsf{GL}(4, \mathbb{C})$,

$$z = x + iy \mapsto \begin{pmatrix} \begin{smallmatrix} \cos(x\varphi) & -\sin(x\varphi) \\ \sin(x\varphi) & \cos(x\varphi) \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \exp(z\log(\lambda)) & 0 \\ 0 & \exp(-z\log(\lambda)) \end{smallmatrix} \end{pmatrix}$$

is a homomorphism. Moreover, $\alpha(1) = A$, and the kernel of α equals $\frac{2\pi i}{\log(\lambda)} \cdot \mathbb{Z}$. Set

$$\Gamma := \mathbb{Z} \oplus \left(\tfrac{2\pi i}{\log(\lambda)} \cdot \mathbb{Z} \right) \subset \mathbb{C}.$$

Recall that there is a lattice $\mathsf{L} \subset \mathbb{R}^4$ that is invariant under A. Obviously, the lattice $\mathsf{L}_{\mathbb{Z}[i]} := \mathsf{L} \oplus i \cdot \mathsf{L} \subset \mathbb{C}^4$ is invariant under $\alpha(\Gamma)$. Consider the semidirect product

$$\mathsf{R} := \mathbb{C}^4 \rtimes_\alpha \mathbb{C}, \quad (v, z) \cdot (w, c) := \big(v + \alpha(z)(w), z + c \big),$$

and the discrete cocompact subgroup $\Lambda := \mathsf{L}_{\mathbb{Z}[i]} \rtimes_\alpha \Gamma$. Similarly to above we can describe R also as a semidirect product $\mathbb{C}^2 \rtimes_\beta \mathsf{R}_1$, where $\mathsf{R}_1 = \big(\{0\} \times \mathbb{C}^2 \big) \rtimes_\alpha \mathbb{C}$ is a complex solvable Lie group, and

$$\beta\colon \mathsf{R}_1 \to \mathsf{O}(2, \mathbb{R}) \subset \mathsf{GL}(2, \mathbb{C})$$

is an unitary representation.

Clearly, we can find a left invariant metric g_1 on R_1 with respect to which (R_1, g_1) is a Kähler manifold. If we denote by $\langle \cdot, \cdot \rangle$ the natural scalar product on $\mathbb{C}^2$, then the product metric $g := \langle \cdot, \cdot \rangle \times g_1$ induces a left invariant Kähler structure on R. Thus $(N, g) := \Lambda \backslash (\mathsf{R}, g)$ is a compact Kähler manifold, and analogously to above one can show that the holonomy group of (N, g) is isomorphic to $\mathsf{U}(3) \times \mathbb{Z}$.

3. The holonomy group is compact if and only if the holonomy representation has finite image

The aim of this section is to prove the following.

Proposition 3.1. *Let M and $\rho_1 : \pi_1(M) \to \mathsf{O}(V)$ be as in Theorem 2. Then the holonomy group of M is compact if and only if the image of ρ_1 is finite.*

Actually this proposition is a direct consequence of the proof of the theorem of Cheeger and Gromoll [4] quoted at the beginning of the introduction. But to avoid mysteries we just repeat the arguments that are necessary:

Proof. If the holonomy group $\mathrm{Hol}(M, p)$ of M at p is compact, then $\mathrm{Hol}(M, p)$ has only finitely many connected components. Taking into account that the identity component $\mathrm{Hol}_0(M, p)$ of $\mathrm{Hol}(M, p)$ acts by definition trivially on $V \subset T_p M$, we see that the image of ρ_1 is finite.

Assume now conversely that the image of ρ_1 is finite. Let $\mathrm{q} \colon \tilde{M} \to M$ be the universal covering map of M, and let $\Pi \subset \mathrm{Iso}(\tilde{M})$ be the deck transformation group. The identity component of the holonomy group of M is isomorphic to the identity component of the holonomy group of $\tilde{M}$ and hence compact. It remains to show that $\mathrm{Hol}(M, p)$ has only finitely many connected components. By de Rahm's decomposition theorem the manifold $\tilde{M}$ splits isometrically as a product

$$\tilde{M} = \tilde{N}_1 \times \cdots \times \tilde{N}_k \times \mathbb{R}^l,$$

where N_i is a manifold with an irreducible holonomy group and $\dim(\tilde{N}_i) \geq 2$; see [2, Theorem 10.43]. The decomposition is unique up to the order of the factors, and for that reason there is a subgroup of finite index in Π that preserves the order. Clearly, we can assume that Π itself preserves the order.

We let $\mathrm{pr}_i\colon \Pi \to \mathrm{Iso}(\tilde{N}_i)$ denote the projection onto the i-th factor. For each i we choose $p_i \in \tilde{N}_i$ and define a group $\Phi_i \subset \mathsf{O}(T_{p_i}\tilde{N}_i)$ consisting of all endomorphisms of the form $\iota_*^{-1} \circ \mathrm{Par}_\gamma\colon T_{p_i}\tilde{N}_i \to T_{p_i}\tilde{N}_i$, where $\iota \in \mathrm{pr}_i(\Pi)$, γ is a piecewise smooth curve from p_i to $\iota(p_i)$, and Par_γ denotes the parallel transport along γ. Notice that Φ_i is isomorphic to the image of the representation of the holonomy group $\mathrm{Hol}(M,p)$ of M that is defined by restricting an element of $\mathrm{Hol}(M,p) \subset \mathsf{O}(T_pM)$ to the subspace corresponding to the i-th factor of $\tilde{M}$. In order to show that $\mathrm{Hol}(M,p)$ has only finitely many connected components, it is therefore sufficient to verify that Φ_i has only finitely many connected components, $i = 1, \dots, k$. The group $\mathrm{Hol}(\tilde{N}_i, p_i)$ is the identity component of Φ_i, and accordingly Φ_i is contained in the normalizer of $\mathrm{Hol}(\tilde{N}_i, p_i)$ in $\mathsf{O}(T_{p_i}\tilde{N}_i)$. The holonomy classification theorem of Berger and Simons can be used to prove that $\mathrm{Hol}(\tilde{N}_i, p_i)$ has finite index in its normalizer unless $\tilde{N}_i$ is Ricci-flat; see [2, p. 308] and the references there. In particular, $\mathrm{Hol}(\tilde{N}_i, p_i)$ has then finite index in Φ_i. Thus we have reduced the situation to case of a Ricci-flat manifold $\tilde{N}_i$.

By construction the isometry group of $\tilde{N}_i$ acts cocompactly, and hence $\tilde{N}_i$ itself is compact, unless it contains a line. The existence of a line would imply via the splitting theorem of Cheeger and Gromoll [4] that the Ricci-flat manifold $\tilde{N}_i$ decomposes as a Riemannian product which is impossible. So $\tilde{N}_i$ is compact. Suppose now that $\mathrm{Hol}(\tilde{N}_i, p_i)$ has infinite index in Φ_i. Then $\mathrm{pr}_i(\Pi) \subset \mathrm{Iso}(\tilde{N}_i)$ is infinite. Since $\mathrm{Iso}(\tilde{N}_i)$ is by [7] a compact Lie group, there must be a nontrivial Killing field on $\tilde{N}_i$. By   Bochner [3, Theorem 2, p. 782] a Killing field on a compact manifold with non-positive Ricci curvature is parallel. But the irreducible manifold $\tilde{N}_i$ admits no nontrivial parallel vectorfields – a contradiction.   q.e.d.

4. Completely reducible representations

In this section we prove several Lemmas on completely reducible representations, which are needed for the proofs of Theorem 2 and Corollary 7.

We recall that a representation $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ is called completely reducible if and only if ρ is the direct sum of irreducible subrepresentations. It is an elementary fact that this is equivalent to saying that for any invariant subspace $X \subset V$ there is an invariant subspace Y satisfying $V = X \oplus Y$.

Lemma 4.1. *Let* $\mathbb{F} \subset \bar{\mathbb{F}}$ *be a field extension,* $\rho\colon \mathsf{G} \to \mathsf{GL}(V)$ *a completely reducible representation in a vector space* V *over* $\bar{\mathbb{F}}$. *Suppose that there is finite dimensional* $\mathbb{F}$-*subspace* $Z \subset V$ *that is invariant under* ρ. *Then the induced* $\mathbb{F}$-*representation in* Z *is completely reducible, too.*

Proof. Without loss of generality there is a nontrivial irreducible $\mathbb{F}$-subspace $Z' \subsetneq Z$. Observe that for $\lambda \in \bar{\mathbb{F}}$ the vector space $\lambda \cdot Z'$ over $\mathbb{F}$ is invariant and irreducible, too. Evidently, the sum of all subspaces $(\lambda \cdot Z')_{\lambda \in \bar{\mathbb{F}}}$ equals $\mathrm{span}_{\bar{\mathbb{F}}}(Z')$. Using Zorn's Lemma one can find numbers $(\lambda_i \in \bar{\mathbb{F}})_{i \in I}$ such that $\mathrm{span}_{\bar{\mathbb{F}}}(Z')$ decomposes as a direct sum, $\mathrm{span}_{\bar{\mathbb{F}}}(Z') = \bigoplus_{i \in I} \lambda_i Z'$. Furthermore, we can assume that $1 \in I$ and $\lambda_1 = 1$. Since the $\bar{\mathbb{F}}$-representation ρ is completely reducible, there is a ρ-invariant subspace U with $V = \mathrm{span}_{\bar{\mathbb{F}}}(Z') \oplus U$. Thus $Z'' := Z \cap \left(U \oplus \bigoplus_{i \in I \setminus \{1\}} \lambda_i Z' \right)$ is an invariant $\mathbb{F}$-subspace and $Z = Z' \oplus Z''$. By induction on $\dim(Z)$ we can assume that the induced $\mathbb{F}$-representation in Z'' is completely reducible. q.e.d.

Lemma 4.2. *Let* $\rho\colon \mathsf{G} \to \mathsf{GL}(n, \mathbb{Q}) \subset \mathsf{GL}(n, \mathbb{C})$ *be a representation. Suppose that there is a real invariant subspace* $U \subset \mathbb{R}^n$ *satisfying the following two conditions.*

(i) *The induced real representation in* U *is completely reducible.*

(ii) *The additive group* $\mathbb{Z}^n + U$ *is dense in* $\mathbb{R}^n$.

Then ρ, *regarded as a complex (real or rational) representation, is completely reducible.*

Proof. Remark that $\mathsf{Z} := \mathrm{span}_{\mathbb{C}}(\rho(\mathsf{G})) \cap \mathsf{GL}(n, \mathbb{C})$ is a Lie group. By definition the space $\mathrm{span}_{\mathbb{C}}(\rho(\mathsf{G}))$ has a complex basis consisting of rational matrices. It follows that Z is an algebraic group defined over $\mathbb{Q}$, i.e., Z is the zero set of a collection of rational polynomials in the coefficients a_{ij}.

Notice that a real subspace $V \subset \mathbb{R}^l$ is invariant under $\rho(\mathsf{G})$ if and only if it is invariant under $\mathsf{Z}_{\mathbb{R}} = \mathsf{Z} \cap \mathsf{GL}(n, \mathbb{R})$. The unipotent radical N of Z is an algebraic group defined over $\mathbb{Q}$, too. Therefore $\mathsf{N} \cap \mathsf{GL}(n, \mathbb{Q})$ is Zarisky dense in N; see [8, p.10]. By Engel's theorem,

$$V := \left\{ v \in \mathbb{R}^n \mid Av = v \text{ for all } A \in \mathsf{N} \cap \mathsf{GL}(n, \mathbb{Q}) \right\} \neq \{0\}.$$

More precisely, we deduce from Engel's theorem that

$$U' := U \cap V \neq \{0\}.$$

The spaces V and U' are ρ-invariant because $\rho(\mathsf{G})$ normalizes $\mathsf{N} \cap \mathsf{GL}(n, \mathbb{Q})$.

Suppose that $U' \neq U$. Since the induced representation in U is completely reducible, there is a ρ-invariant subspace U'' with

$$U = U' \oplus U''.$$

The space U'' is also invariant under $\mathsf{Z}_{\mathbb{R}} \supset \mathsf{N} \cap \mathsf{GL}(n, \mathbb{Q})$. Once again Engel's theorem applies, so $V \cap U'' \neq \{0\}$ which is impossible.

This proves $U \subset V$. From the definition of V it is clear that $V \cap \mathbb{Z}^n$ is a lattice in V, and accordingly $V + \mathbb{Z}^n$ is closed. On the other hand, $V + \mathbb{Z}^n \supset U + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$ and thus $V = \mathbb{R}^n$. In other words, $\mathsf{N} \cap \mathsf{GL}(n, \mathbb{Q})$ is the trivial group. Hence the unipotent radical N of the algebraic group Z is trivial, and for that reason the natural representation of Z in $\mathbb{C}^n$ is completely reducible; see [8, p.11]. Now, let $X \subset \mathbb{C}^n$ be a complex ρ-invariant subspace. Since X is also invariant under Z, there is a Z-invariant complex subspace Y satisfying $\mathbb{C}^n = X \oplus Y$. Evidently, Y is ρ-invariant. Consequently, ρ is as a complex representation completely reducible. By Lemma 4.1, ρ is also as a real (rational) representation completely reducible.   q.e.d.

Lemma 4.3. *Let* $\rho \colon \mathsf{G} \to \mathsf{GL}(n, \mathbb{Q}) \subset \mathsf{GL}(n, \mathbb{R})$ *be a representation. Suppose that* ρ *is completely reducible as a rational representation. Then* ρ *is completely reducible as a real representation.*

Proof. Choose an irreducible real subspace $U \subset \mathbb{R}^n$ with $U \neq \{0\}$. Let W be the identity component of the closure of $U + \mathbb{Z}^n$. For $g \in \mathsf{G}$ the group $\mathbb{Z}^n \cap \big(\rho(g)(\mathbb{Z}^n)\big)$ has finite index in both $\mathbb{Z}^n$ and $\rho(g)(\mathbb{Z}^n)$. Hence W is also the identity component of the closure of $\rho(g)(\mathbb{Z}^n) + U$. Consequently, W is a ρ-invariant real subspace. Moreover, $W_{\mathbb{Z}} = W \cap \mathbb{Z}^n$ is a lattice in W. Since $W_{\mathbb{Z}} + U$ is dense in W, we deduce from Lemma 4.2 that the induced real representation in W is completely reducible. By assumption there is a ρ-invariant subspace $V_{\mathbb{Q}} \subset \mathbb{Q}^n$ such that $\mathbb{Q}^n = V_{\mathbb{Q}} \oplus (\mathbb{Q}^n \cap W)$. Set $V = \mathrm{span}_{\mathbb{R}}(V_{\mathbb{Q}})$. Clearly, V is an invariant subspace of ρ and $\mathbb{R}^n = V \oplus W$. By induction on n we can assume that the induced real representation in V is completely reducible, too.   q.e.d.

Lemma 4.4. *Let* $\rho \colon \mathsf{G} \to \mathsf{GL}(V)$ *be a completely reducible real representation. Suppose there is a finitely generated, ρ-invariant subgroup* $\Lambda \subset V$. *Then for any ρ-invariant subgroup* $\Lambda' \subset \Lambda$ *there is a ρ-invariant subgroup* Λ'' *such that* $\Lambda' \cap \Lambda'' = \{0\}$, *and* $\Lambda' \oplus \Lambda''$ *is a subgroup of finite index in* Λ.

Proof. By Lemma 4.1 the induced rational representation in $\Lambda_{\mathbb{Q}} :=$ $\mathrm{span}_{\mathbb{Q}}(\Lambda)$ is completely reducible. Since $\Lambda'_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}(\Lambda')$ is a ρ-invariant subspace of $\Lambda_{\mathbb{Q}}$, there is a ρ-invariant subspace $Y \subset \Lambda_{\mathbb{Q}}$ such that $\Lambda_{\mathbb{Q}} = Y \oplus \Lambda'_{\mathbb{Q}}$. Therefore $\Lambda'' := Y \cap \Lambda$ is a group satisfying the conclusion of the lemma.   q.e.d.

Lemma 4.5. *Let $\rho \colon \mathsf{G} \to \mathsf{GL}(V)$ be a real representation. Suppose there is a finitely generated, ρ-invariant subgroup $\Lambda \subset V$ for which* $\mathrm{span}_{\mathbb{R}}(\Lambda) = V$. *Then for any element $v \in V$ the group*

$$\langle \mathsf{G} \star v \rangle_{\mathbb{Z}} := \mathrm{span}_{\mathbb{Z}}\{\rho(g)(v) \mid g \in \mathsf{G}\}$$

is finitely generated.

Proof.   Choose $a_1, \ldots, a_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in \Lambda$ such that $\sum_{i=1}^{n} a_i v_i = v$. Then $\langle \mathsf{G} \star v \rangle_{\mathbb{Z}}$ is contained in the finitely generated, free abelian group $a_1 \Lambda + \cdots + a_n \Lambda$, and thus $\langle \mathsf{G} \star v \rangle_{\mathbb{Z}}$ is finitely generated, too.   q.e.d.

Lemma 4.6. *Let $\rho \colon \mathsf{G} \to \mathsf{GL}(V)$ be a completely reducible real representation, and let $\Lambda \subset V$ be a finitely generated ρ-invariant cocompact subgroup. Then there is a ρ-invariant subgroup $\Lambda' \subset \Lambda$ satisfying:*

(i) $\mathrm{rank}(\Lambda') = \mathbb{Z}\text{-rk}(\rho)$; *see equation* (1) *for the definition of* $\mathbb{Z}\text{-rk}(\rho)$,

(ii) *the induced representation $\tilde{\rho} : \mathsf{G} \to \mathsf{GL}(\Lambda' \otimes_{\mathbb{Z}} \mathbb{R})$ decomposes as a direct sum $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$, where $\tilde{\rho}_1$ is equivalent to ρ.*

Proof.   Let $Z \subset V$ be a ρ-invariant cocompact subgroup with $\mathrm{rank}(Z) = \mathbb{Z}\text{-rk}(\rho)$, and set $W := \mathrm{span}_{\mathbb{Q}}(\Lambda)$ and $Z_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}(Z)$. We claim that there is a ρ-invariant subspace $W' \subset W$ such that the induced rational representation in W' is equivalent to the induced rational representation in $Z_{\mathbb{Q}}$.

By Lemma 4.1 there is a decomposition $Z_{\mathbb{Q}} = Z_1 \oplus \cdots \oplus Z_k$ into nontrivial irreducible subspaces. Clearly, there are numbers $\lambda_1, \cdots, \lambda_q \in \mathbb{R}$ such that $W \subset \lambda_1 Z_{\mathbb{Q}} \oplus \cdots \oplus \lambda_q Z_{\mathbb{Q}}$. It follows that any irreducible subrepresentation of W is equivalent to an irreducible subrepresentation of Z.

If the induced rational representations in the subspaces $Z_1, \cdots, Z_k$ are pairwise equivalent, then we can argue as follows. Let $W_1 \oplus \cdots \oplus W_l$ be a decomposition into nontrivial irreducible ρ-invariant subspaces. Clearly, the induced representation in W_i is equivalent to the induced representation in Z_j for all i, j. Since Λ is cocompact in V, we obtain

$\dim_{\mathbb{Q}}(W) \geq \mathbb{Z}\text{-rk}(\rho)$ and accordingly $l \geq k$. Thus we can define $W' = W_1 \oplus \cdots \oplus W_k$.

If the induced rational representations in the subspaces $Z_1, \cdots, Z_k$ are not pairwise equivalent, then after reordering we can assume that for some positive integer $k' \leq k - 1$ the induced representation in Z_i is equivalent the one in Z_1 if and only if $i \leq k'$. Set

$$V' := \operatorname{span}_{\mathbb{R}}(Z_1 \oplus \cdots \oplus Z_{k'})$$

and

$$V'' := \operatorname{span}_{\mathbb{R}}(Z_{k'+1} \oplus \cdots \oplus Z_k).$$

Clearly, $V = V' \oplus V''$ and it is straightforward to check that $W = (W \cap V') \oplus (W \cap V'')$. The statement follows by induction on $\dim(V)$.

Hence we can choose a ρ-invariant $\mathbb{Q}$-subspace $W' \subset W$ such that the induced representations in W' and $Z_{\mathbb{Q}}$ are equivalent. Set $\Lambda' := W' \cap \Lambda$. Then $\operatorname{rank}(\Lambda') = \dim_{\mathbb{Q}}(W') = \mathbb{Z}\text{-rk}(\rho_1)$. Moreover, the natural representation of G in $\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to the natural representation in $Z_{\mathbb{Q}}$. Consequently, the natural representation of G in $\Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$ is equivalent to the representation $\bar{\rho}$ of G in $Z \otimes_{\mathbb{Z}} \mathbb{R}$. By Lemma 4.3 the representation $\bar{\rho}$ is completely reducible. The inclusion $Z \to V$ induces an equivariant epimorphism $Z \otimes_{\mathbb{Z}} \mathbb{R} \to V$. Thus the completely reducible representation $\bar{\rho}$ decomposes as a direct sum $\bar{\rho} = \bar{\rho}_1 \oplus \bar{\rho}_2$, where $\bar{\rho}_1$ is equivalent to ρ. Since $\bar{\rho}$ is equivalent to $\tilde{\rho}$, this completes the proof of the lemma.   q.e.d.

5. Algebraic properties of the holonomy representation

5.1   The proof of Theorem 2

The universal covering space $\tilde{M}$ of M is isometric to a Riemannian product $\tilde{M} = \tilde{N} \times \mathbb{R}^l$, where $\tilde{N}$ is a simply connected manifold without Euclidean factor. Let $\Pi \subset \operatorname{Iso}(\tilde{N}) \times \operatorname{Iso}(\mathbb{R}^l) = \operatorname{Iso}(\tilde{M})$ be the deck transformation group. Clearly, $\Pi \cong \pi_1(M)$, and M is isometric to the orbit manifold $\tilde{N} \times \mathbb{R}^l / \Pi$. Furthermore, under the identification $\pi_1(M) = \Pi$ the holonomy representation ρ_1 is equivalent to $\psi_{|\Pi}$, where

$$(3) \qquad \psi \colon \operatorname{Iso}(\tilde{N}) \times \operatorname{Iso}(\mathbb{R}^l) \to \operatorname{Iso}(\mathbb{R}^l) = \mathbb{R}^l \rtimes \mathsf{O}(l) \to \mathsf{O}(l)$$

is the projection. In the following we identify $\mathbb{R}^l$ and $\mathsf{O}(l)$ with the subgroups $\mathbb{R}^l \times \{e\}$ and $\{0\} \times \mathsf{O}(l)$ of $\operatorname{Iso}(\mathbb{R}^l) = \mathbb{R}^l \rtimes \mathsf{O}(l)$, and we view

$\text{Iso}(\mathbb{R}^l)$ and $\text{Iso}(\tilde{N})$ in the natural fashion as subgroups of $\text{Iso}(\tilde{N}) \times \text{Iso}(\mathbb{R}^l) = \text{Iso}(\tilde{M})$. Recall that, by a theorem of [7], $\text{Iso}(\tilde{M})$ is a Lie group. Since Π acts discontinuously and cocompactly on $\tilde{M}$, it follows that $\Pi \subset \text{Iso}(\tilde{M})$ is a discrete, cocompact subgroup. Denote by $\text{Iso}_0(\tilde{M})$ the identity component of $\text{Iso}(\tilde{M})$. Then $\Pi^* := \Pi \cap \text{Iso}_0(\tilde{M})$ is a discrete, cocompact subgroup of $\text{Iso}_0(\tilde{M})$.

Consider the projection $\text{pr} \colon \text{Iso}_0(\tilde{M}) \to \text{Iso}_0(\tilde{M})/\mathbb{R}^l$, and let A be the identity component of the closure of $\text{pr}(\Pi^*)$. The group A is solvable because the kernel of pr is abelian; see [8, Theorem 8.24]. Evidently, $\mathsf{B} := \text{pr}^{-1}(\mathsf{A})$ is also solvable and Π normalizes B. By construction the group $\Pi \cdot \mathsf{B}$ is the closure of $\Pi \cdot \mathbb{R}^n$. Taking into account that Π is cocompact in $\text{Iso}(\tilde{M})$, we see that $\Pi \cap \mathsf{B}$ is a cocompact subgroup of B.

Let N be the maximal connected, nilpotent normal subgroup of B. By a theorem of Mostov $\Pi \cap \mathsf{N}$ is a cocompact subgroup of N, and thus $\Pi \cdot \mathsf{N} \subset \text{Iso}(\tilde{M})$ is closed; see [8, Theorem 3.3.]. Furthermore, $\mathbb{R}^l \subset \mathsf{N}$ and accordingly $\mathsf{B} = \mathsf{N}$.

Since B is nilpotent, the representation of B in $\mathbb{R}^l$ induced by conjugation is unipotent. On the other hand, this representation coincides with $\psi_{|\mathsf{B}}$, where ψ is defined in equation (3). The image $\psi(\mathsf{B})$, being both unipotent and orthogonal, must be trivial, and therefore $\mathbb{R}^l$ is contained in the center of B.

Next we claim that B is abelian. Otherwise the dense subgroup $(\Pi \cdot \mathbb{R}^l) \cap \mathsf{B}$ of B would contain two sequences a_i, b_i tending to e such that a_i does not commute with b_i. The commutator sequence $c_i := [a_i, b_i] \neq e$ converges to e, too. But $\mathbb{R}^n$ is contained in the center of B, and accordingly we can write c_i also as a commutator of elements in Π. In particular, $c_i \in \Pi$ which is impossible.

Hence B is abelian, and $\mathsf{D} := \mathsf{B} \cap \Pi$ is a discrete, cocompact subgroup of B. Notice that the exponential map $\exp \colon \mathfrak{b} \to \mathsf{B}$ is a homomorphism and that $\Lambda := \exp^{-1}(\mathsf{D})$ is a lattice in $\mathfrak{b}$. We identify $\mathbb{R}^l \subset \mathsf{B}$ with its Lie algebra and let $\mathfrak{w} \subset \mathfrak{b}$ denote the Lie algebra of $\mathsf{B} \cap \text{Iso}(\tilde{N})$. Since B is the closure of $\mathbb{R}^l \cdot \mathsf{D}$, it follows that $\mathfrak{b}$ is the closure of $\mathbb{R}^l + \Lambda$. The adjoint map of $\Pi \cdot \mathsf{B}$ induces a representation

$$\bar{\rho} \colon \Pi \to \mathsf{GL}(\mathfrak{b}).$$

Clearly, $\mathbb{R}^l$ and $\mathfrak{w}$ are $\bar{\rho}$-invariant and the induced representation in $\mathbb{R}^l$ is the orthogonal representation $\psi_{|\Pi}$. Moreover, $\Lambda + \mathbb{R}^l$ is dense in $\mathfrak{b} = \mathbb{R}^l \oplus \mathfrak{w}$, and the lattice Λ is invariant under $\bar{\rho}$. Via Lemma 4.2 this implies that $\bar{\rho}$ is completely reducible.

Observe that $\mathfrak{w}_\Lambda = \Lambda \cap \mathfrak{w}$ is invariant under $\bar{\rho}$. By Lemma 4.4 there is a $\bar{\rho}$-invariant subgroup $\hat{\mathsf{L}} \subset \Lambda$ satisfying $\hat{\mathsf{L}} \cap \mathfrak{w}_\Lambda = \{0\}$ and $\Lambda' = \hat{\mathsf{L}} \oplus \mathfrak{w}_\Lambda$ is a subgroup of finite index in Λ. Let $f \colon \mathfrak{b} \to \mathbb{R}^l$ be the projection with kernel $\mathfrak{w}$. Clearly, f is an equivariant homomorphism from $(\mathfrak{b}, \bar{\rho})$ onto $(\mathbb{R}^l, \psi_{|\Pi})$, and the restriction $f_{|\hat{\mathsf{L}}}$ is injective. The image $f(\hat{\mathsf{L}})$ is of finite index in $f(\Lambda)$. In particular, $f(\hat{\mathsf{L}})$ is cocompact in $\mathbb{R}^l$. By Lemma 4.6 there is a subgroup $\mathsf{L} \subset \hat{\mathsf{L}} \cong f(\hat{\mathsf{L}})$, such that

$$\mathrm{rank}(\mathsf{L}) = \mathbb{Z}\text{-rk}(\psi_{|\Pi}) = \mathbb{Z}\text{-rk}(\rho_1)$$

and the induced real representation $\tilde{\rho}$ in $\mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R}$ decomposes as a direct sum $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$, where $\tilde{\rho}_1$ is equivalent to $\psi_{|\Pi} \cong \rho_1$.

The space $\mathsf{L}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(\mathsf{L})$ has trivial intersection with

$$\mathrm{span}_{\mathbb{R}}(\mathrm{Ker}(\exp_{|\mathfrak{b}})) \subset \mathrm{span}_{\mathbb{R}}(\mathfrak{w}_\Lambda).$$

Thus $\exp_{|\mathsf{L}_{\mathbb{R}}}$ is an embedding and we can identify $\mathsf{L}_{\mathbb{R}}$ with the closed subgroup $\exp(\mathsf{L}_{\mathbb{R}}) \subset \mathsf{B}$. Notice that L becomes under this identification a normal subgroup of Π. For later applications we remark that Π normalizes the simply connected, closed, abelian group $\mathsf{L}_{\mathbb{R}} \subset \mathsf{B}$.

It remains to verify statement (iii). The group $\mathsf{L} \subset \mathsf{B}$ has trivial intersection with $\mathsf{B} \cap \mathrm{Iso}(\tilde{N})$. Therefore the projection

$$h \colon \mathrm{Iso}(\tilde{N}) \times \mathrm{Iso}(\mathbb{R}^l) \to \mathrm{Iso}(\mathbb{R}^l)$$

maps L injectively onto a subgroup of $\mathbb{R}^l \subset \mathrm{Iso}(\mathbb{R}^l)$.

Since M is compact, its fundamental group $\pi_1(M) = \Pi$ is finitely generated. Let $b_1, \ldots, b_q \in \Pi$ be elements that generate Π, and let

$$(v_i, A_i) := h(b_i) \in \mathbb{R}^l \rtimes \mathsf{O}(l) = \mathrm{Iso}(\mathbb{R}^l), \quad i = 1, \ldots, q.$$

The finitely generated cocompact subgroup $f(\hat{\mathsf{L}})$ of $\mathbb{R}^l$ is invariant under $\psi(\Pi) \subset \mathsf{O}(l)$. Hence we can employ Lemma 4.5 in order to see that the group

$$\mathsf{C} := \mathrm{span}_{\mathbb{Z}}\big\{\psi(g)(v_i) \mid g \in \Pi, \ i = 1, \ldots, q\big\} \subset \mathbb{R}^l$$

is finitely generated. Evidently, $h(\Pi)$ normalizes C, and the product $h(\Pi) \cdot \mathsf{C}$ contains the elements $A_1, \ldots, A_q \in \mathsf{O}(l) \subset \mathrm{Iso}(\mathbb{R}^l)$ which generate $\psi(\Pi)$. Therefore

$$h(\Pi) \cdot \mathsf{C} = \psi(\Pi) \cdot \mathsf{C}.$$

Next, we remark that $h(\mathsf{L}) \subset \mathsf{C}$ is invariant under $\psi(\Pi)$. By Lemma 4.4 there is a $\psi(\Pi)$-invariant subgroup $\mathsf{C}' \subset \mathsf{C}$ such that $\mathsf{C}' \oplus h(\mathsf{L})$ has finite index in C. For the group $\bar{\mathsf{H}} := (\psi(\Pi) \cdot \mathsf{C}') \cap h(\Pi)$ this implies that $\bar{\mathsf{H}} \cap h(\mathsf{L}) = \{e\}$ and that

$$\bar{\mathsf{H}} \cdot h(\mathsf{L}) = h(\Pi) \cap \big(\psi(\Pi) \cdot (\mathsf{C}' \oplus h(\mathsf{L}))\big)$$

is of finite index in $h(\Pi)$. Thereby the group $\mathsf{H} := h_{|\Pi}^{-1}(\bar{\mathsf{H}})$ satisfies (iii).

5.2 The proof of Corollary 3

We use the notations of Theorem 2. By Proposition 3.1 the image of ρ_1 (and of $\tilde{\rho}_1$) is infinite.

Suppose for a moment that the image of $\tilde{\rho}_1$ contains only elements of finite order. Then any eigenvalue of an element in

$$\tilde{\rho}_1(\pi_1(M)) \cong \rho_1(\pi_1(M)) \subset \mathsf{O}(V)$$

is a root of unity. Taking into account that the representation $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$ leaves a lattice invariant, we see that the degree of the field extension $\mathbb{Q} \subset \mathbb{Q}(\lambda)$ is bounded by $\mathrm{rank}(\mathsf{L})$ for any eigenvalue λ of an element in $\tilde{\rho}_1(\pi_1(M))$. But then the order of any element in $\tilde{\rho}_1(\pi_1(M))$ is bounded by a constant only depending on $\mathrm{rank}(\mathsf{L})$. Consequently, the relatively compact group $\tilde{\rho}_1(\pi_1(M))$ is discrete and hence finite – a contradiction.

Thus for some $g \in \pi_1(M)$ the semisimple endomorphism $\tilde{\rho}_1(g)$ has infinite order. Since the eigenvalues of $\tilde{\rho}_1(g)$ have absolute value 1, there must be an eigenvalue $\lambda \in \mathsf{S}^1 \subset \mathbb{C}$ of $\tilde{\rho}(g) = \tilde{\rho}_1(g) \oplus \tilde{\rho}_2(g)$ that is not a root of unity. By the very definition of $\tilde{\rho}$ the number λ is also an eigenvalue of

$$c_g \colon \mathsf{L} \to \mathsf{L}, \ v \mapsto gvg^{-1}.$$

It remains to verify the inequalities

$$\mathrm{rank}(\mathsf{L}) \geq \dim(V) + 2 \geq 4.$$

Using that $\rho_1(\pi_1(M)) \subset \mathsf{O}(V)$ is infinite we obtain $\dim(V) \geq 2$.

Suppose, on the contrary, that

$$\dim(V) + 1 \geq \mathrm{rank}(\mathsf{L}) = \dim(\mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R}).$$

Then the rank of the representation $\tilde{\rho}_2$ is at most one. The representation $\tilde{\rho}_1$ is equivalent to the orthogonal representation ρ_1, and $\tilde{\rho}$ leaves the lattice $\mathsf{L} \subset \mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R}$ invariant. In particular, $|\det(\tilde{\rho}_1)| = |\det(\tilde{\rho})| \equiv 1$.

In the case $\mathrm{rank}(\tilde{\rho}_2) = 1$ this implies $|\det(\tilde{\rho}_2)| \equiv 1$, and $\tilde{\rho}_2$ is then orthogonal, too. For that reason $\tilde{\rho}$ itself is equivalent to an orthogonal representation. Thus the image of $\tilde{\rho}$ is relatively compact. Furthermore,

$$\tilde{\rho}(\Pi) \subset \mathsf{GL}(\mathsf{L}) \subset \mathsf{GL}(\mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R})$$

is discrete and hence finite – a contradiction.

6. Implications of a nontrivial holonomy representation

6.1   The proof of Corollary 4

We use the notations and conventions of the proof of Theorem 2. In particular, $\tilde{M} = \tilde{N} \times \mathbb{R}^l$ is the universal covering space of M, and $\pi_1(M) = \Pi \subset \mathrm{Iso}(\tilde{N}) \times \mathrm{Iso}(\mathbb{R}^l)$ is the deck transformation group. Recall that the identity component B of the closure of the group $\mathbb{R}^l \cdot \Pi$ is abelian. The group L constructed in the proof is a subgroup of B. Moreover, there is a connected, simply connected, closed, abelian subgroup $\mathsf{L}_{\mathbb{R}} \subset \mathsf{B}$ that is normalized by Π and that contains L as a discrete, cocompact subgroup. Finally, we can choose a group $\mathsf{H} \subset \Pi$ such that $\mathsf{H} \cap \mathsf{L} = \{0\}$ and $\hat{\Pi} := \mathsf{H} \cdot \mathsf{L}$ has finite index in Π.

In order to prove the addition a) of the corollary later on, it is convenient to assume that the following hold: If the holonomy group of M is compact, then the holonomy group of the finite cover $\tilde{M}/\hat{\Pi}$ is connected. As we know a compact holonomy group to have only finitely many connected components, this assumption can be made without loss of generality.

We proceed with the general case. Remark that $\hat{\Pi}$ is a cocompact subgroup of $\hat{\Pi} \cdot \mathsf{L}_{\mathbb{R}} = \mathsf{H} \cdot \mathsf{L}_{\mathbb{R}}$. Since the orbits of $\hat{\Pi}$ are closed, the orbits of $\mathsf{H} \cdot \mathsf{L}_{\mathbb{R}}$ are closed, too. Next we want to show that $\mathsf{L}_{\mathbb{R}}$ acts freely on $\tilde{M}$. For an arbitrary point $p \in \tilde{M}$ we consider the map $g \colon \mathsf{L}_{\mathbb{R}} \to \tilde{M}$, $v \mapsto v \star p$. Using that g maps the cocompact subgroup $\mathsf{L} \subset \mathsf{L}_{\mathbb{R}}$ monomorphically onto a discrete subset of $\tilde{M}$, it is straightforward to check that g is a proper map. Accordingly the isotropy group $\mathsf{L}_{\mathbb{R}}^p \subset \mathsf{L}_{\mathbb{R}}$ of p is compact. Taking into account that $\mathsf{L}_{\mathbb{R}}$ is a vector group, we see that $\mathsf{L}_{\mathbb{R}}^p = \{e\}$.

Thus the action of $\mathsf{L}_{\mathbb{R}}$ on $\tilde{M}$ is free and the corresponding orbits are closed. Consequently, the orbit space $\tilde{B} := \tilde{M}/\mathsf{L}_{\mathbb{R}}$ is a complete Riemannian manifold. As $\mathsf{L}_{\mathbb{R}}$ is normal in $\mathsf{H} \cdot \mathsf{L}_{\mathbb{R}}$, the group $\mathsf{H} \cong \mathsf{H} \cdot \mathsf{L}_{\mathbb{R}}/\mathsf{L}_{\mathbb{R}}$ acts isometrically and discontinuously on $\tilde{B}$.

Suppose that this action is not free. Then we can find a point $\bar{p} := \mathsf{L}_\mathbb{R} \star p \in \tilde{B}$ for which the isotropy group $\mathsf{H}_{\bar{p}}$ is nontrivial. Since the action of H on $\tilde{B}$ is discontinuous, it follows that $\mathsf{H}_{\bar{p}}$ is finite. The group $\mathsf{H}_{\bar{p}} \subset \mathsf{H} \subset \Pi$ acts freely on the submanifold $\mathsf{L}_\mathbb{R} \star p \subset \tilde{M}$. But this contradicts the fact that $\mathsf{L}_\mathbb{R} \star p$ is isometric to a Euclidean space because a nontrivial finite group can not act freely on $\mathbb{R}^d$.

We have proved that H acts freely and discontinuously on $\tilde{B}$. Therefore $\hat{M} := \tilde{M}/\hat{\Pi}$ fibers over $B = \tilde{B}/\mathsf{H} = \tilde{M}/(\hat{\Pi} \cdot \mathsf{L}_\mathbb{R})$. Clearly, the fiber bundle map $\sigma \colon \hat{M} \to B$ is a Riemannian submersion and the fibers are flat tori diffeomorphic to $\mathsf{L}_\mathbb{R}/\mathsf{L}$. Together with the equality, $\dim(\mathsf{L}_\mathbb{R}) = \operatorname{rank}(\mathsf{L}) = \mathbb{Z}\text{-rk}(\rho_1) \geq \dim(V)$, this establishes the first part of the corollary.

In order to prove the existence of a section, we remark that the manifold $\tilde{M}/\mathsf{H}$ fibers over $B = \tilde{M}/\mathsf{H} \cdot \mathsf{L}_\mathbb{R}$ as well. The fibers of this fibration are diffeomorphic to the vector group $\mathsf{L}_\mathbb{R}$. Hence there is a smooth section $\tilde{s} \colon B \to \tilde{M}/\mathsf{H}$; see [10, Theorem 12.2]. Evidently, $\tilde{s}$ induces a smooth section $s \colon B \to \hat{M} = \tilde{M}/\mathsf{H} \cdot \mathsf{L}$.

If the holonomy group of M is not compact, then $\operatorname{rank}(\mathsf{L}) \geq \dim(V) + 2$; see Corollary 3. Hence we have verified statement b).

Now we want to prove a). So we assume that the holonomy group of M is compact. By construction the holonomy group of the finite cover $\hat{M}$ is then connected. In particular, the holonomy representation of $\pi_1(\hat{M}) \cong \hat{\Pi}$ has trivial image. This implies that the deck transformation group $\hat{\Pi} = \mathsf{H} \cdot \mathsf{L}$ commutes with $\mathbb{R}^l \subset \operatorname{Iso}(\tilde{N}) \times \operatorname{Iso}(\mathbb{R}^l)$. Recall from the proof of Theorem 2 that the natural projection $h \colon \operatorname{Iso}(\tilde{N}) \times \operatorname{Iso}(\mathbb{R}^l) \to \operatorname{Iso}(\mathbb{R}^l)$ maps L injectively onto a subgroup of $\mathbb{R}^l$. Moreover, $h(gag^{-1}) = gh(a)g^{-1} = h(a)$ for $g \in \Pi$ and $a \in \mathsf{L}$. Consequently, $\hat{\Pi}$ commutes with L. Since L is a lattice in the vector group $\mathsf{L}_\mathbb{R}$, it follows that $\hat{\Pi}$ commutes with $\mathsf{L}_\mathbb{R}$ as well.

Thus the torus $\mathsf{T} := \mathsf{L}_\mathbb{R}/\mathsf{L}$ acts freely and isometrically on $\hat{M} = \tilde{M}/\hat{\Pi}$. Furthermore, the orbits of this action are precisely the fibers of the submersion $\sigma \colon \hat{M} \to B$. Therefore $\hat{M}$ is the total space of principal T-bundle over B. Recall that there is a smooth section $s \colon B \to \hat{M} = \tilde{M}/\hat{\Pi}$. Hence the product bundle $(B \times \mathsf{T}, B, \mathsf{T})$ is via

$$\iota \colon B \times \mathsf{T} \to \hat{M}, \quad (b, h) \mapsto h \star s(b)$$

isomorphic to $(\hat{M}, B, \mathsf{T}, \sigma)$.

Next we want to prove statement c). Thus we assume that the Ricci curvature of M is nonpositive, $\operatorname{Ric} \leq 0$. Clearly, it is sufficient to show

that the horizontal distribution of the Riemannian submersion

$$\tilde{\sigma}\colon \tilde{M} \to \tilde{B} = \tilde{M}/\mathsf{L}_\mathbb{R}, \;\; q \mapsto \mathsf{L}_\mathbb{R} \star q$$

is integrable. For $w \in T_q\tilde{M}$ we denote the vertical (resp. horizontal) component of w by w^v (resp. by w^h). Let $v_1,\dots,v_d$ be a basis of the vector group $\mathsf{L}_\mathbb{R} \cong \mathbb{R}^d$, and let $Y_1,\dots,Y_d$ be the corresponding Killing fields on $\tilde{M}$. We claim that the function

$$f\colon \tilde{M} \to \mathbb{R}, \;\; q \mapsto \log\big(\det(\langle Y_i, Y_j \rangle)\big)(q)$$

is subharmonic. Remark that f is up to an additive constant independent of the choice of the basis $v_1,\dots,v_d$. In order to compute $\Delta f(q)$ we can assume that $Y_{1|q},\dots,Y_{d|q}$ are orthonormal vectors. Choose orthonormal vectorfields $e_1,\dots,e_m$ in a neighborhood of q satisfying $\nabla_{e_{i|q}} e_j = 0$ for $i,j = 1,\dots,m$ and $e_{i|q} = Y_{i|q}$ for $i = 1,\dots,d$. Since $Y_{j \circ \exp(te_i)}$ is a Jacobi field along the geodesic $\exp(te_i)$, we obtain

$$\nabla_{e_{i|q}} \nabla_{e_i} Y_j \;=\; -\operatorname{R}(Y_j, e_i) e_{i|q}.$$

Therefore

$$
\begin{aligned}
\Delta f(q) \;&=\; \sum_{k=1}^{m} e_{k|q}\big(e_k \log(\det(\langle Y_i, Y_j \rangle)))\big) \\[4pt]
&=\; \sum_{k=1}^{m} e_{k|q} \operatorname{tr}\big((\langle Y_i, Y_j \rangle)^{-1}(e_k \langle Y_i, Y_j \rangle)\big) \\[4pt]
&=\; \Delta \operatorname{tr}(\langle Y_i, Y_j \rangle)(q) \;-\; \sum_{k=1}^{m} \operatorname{tr}\big((e_k \langle Y_i, Y_j \rangle)^2\big) \\[4pt]
&=\; 2\sum_{k=1}^{m}\sum_{j=1}^{d}\Big(\langle \nabla_{e_{k|q}} \nabla_{e_k} Y_j, Y_j \rangle + \|\nabla_{e_{k|q}} Y_j\|^2\Big) \\[4pt]
&\quad\; -4 \sum_{k,i,j}(\langle \nabla_{e_{k|q}} Y_i, Y_j \rangle)^2 \\[4pt]
&=\; -2 \sum_{k,j}\langle \operatorname{R}(e_k, Y_j)Y_j, e_k \rangle(q) \;+\; 2\sum_{k,j}\|\nabla_{e_{k|q}} Y_j\|^2 \\[4pt]
&\quad\; -2 \sum_{k,i,j}(\langle \nabla_{Y_j} Y_i, e_{k|q} \rangle)^2 \;-\; 2\sum_{k,i}\|(\nabla_{e_{k|q}} Y_i)^v\|^2
\end{aligned}
$$

(4)

$$
\begin{aligned}
&= \ -2 \sum_{j} \mathrm{Ric}(Y_j, Y_j)(q) + 2 \sum_{k,j} \|(\nabla_{e_{k|q}} Y_j)^h\|^2 \\
&\quad -2 \sum_{i,j} \|\nabla_{Y_{j|q}} Y_i\|^2 \\
&= \ -2 \sum_{j=1}^{d} \mathrm{Ric}(Y_j, Y_j)(q) \ + \ 2 \sum_{k=d+1}^{m} \sum_{j=1}^{d} \|(\nabla_{e_{k|q}} Y_j)^h\|^2 \\
&\geq \ 0.
\end{aligned}
$$

For $g \in \Pi$ the endomorphism $c_g \colon \mathsf{L}_{\mathbb{R}} \to \mathsf{L}_{\mathbb{R}}, v \mapsto gvg^{-1}$ leaves the lattice $\mathsf{L} \subset \mathsf{L}_{\mathbb{R}}$ invariant. Hence, $|\det(c_g)| = 1$. With this in mind it is straightforward to check that f is invariant under the deck transformation group Π, that is $f(q) = f(g \star q)$ for all $g \in \Pi$ and $q \in \tilde{M}$. Therefore the subharmonic function f attains its maximum on a compact fundamental domain, and by the maximum principle f is constant. In particular, equality holds in inequality (5). Consequently, the vertical distribution is parallel along a horizontal geodesic. This clearly implies that the horizontal distribution is parallel along a horizontal geodesic, too. For that reason the Lie bracket $[H_1, H_2] = \nabla_{H_1} H_2 - \nabla_{H_2} H_1$ of two horizontal vectorfields is horizontal, too. Thus the horizontal distribution is integrable.

Next we want to prove the addition d). First notice that the proof of c) yields another result: Since the function f is constant, the mean curvature of all fibers of $\tilde{\sigma}$ vanishes. For $q \in \tilde{M}$ we choose as above vertical Killing fields $Y_1, \dots, Y_d$ induced by the action of $\mathsf{L}_{\mathbb{R}}$ such that $Y_{1|q}, \dots, Y_{d|q}$ are orthonormal. Clearly, the vertical component of $\nabla_{Y_i} Y_j$ vanishes. Because of the vanishing mean curvature we obtain $\sum_{i=1}^{d} \nabla_{Y_{i|q}} Y_i = 0$. Using the Gauss equations we compute the (intrinsic) scalar curvature $\mathrm{scal}_{\mathrm{fib}}(q)$ of the fiber $\mathsf{L}_{\mathbb{R}} \star q$ in q

$$
(5) \qquad \mathrm{scal}_{\mathrm{fib}}(q) \ = \ \sum_{\substack{i,j=1 \\ i \neq j}}^{d} \mathrm{K}\big(\mathrm{span}_{\mathbb{R}}(Y_{i|q}, Y_{j|q})\big) - \sum_{i,j=1}^{d} \|\nabla_{Y_{i|q}} Y_j\|^2,
$$

where K denotes the sectional curvature in $\tilde{M}$ – a function which by assumption is nonpositive. On the other hand, the submanifold $\mathsf{L}_{\mathbb{R}} \star q$ is isometric to $\mathbb{R}^d$ and accordingly $\mathrm{scal}_{\mathrm{fib}}(q) = 0$. So by equation (5) the second fundamental form of $\mathsf{L}_{\mathbb{R}} \star q$ in q vanishes. Hence the fibers of σ are totally geodesic.

In summary we can say that the horizontal distribution of $\tilde{\sigma}$ is integrable and that the fibers $\tilde{\sigma}$ are totally geodesic. Since $\tilde{B}$ is simply

connected, it follows that $\tilde{M}$ is isometric to a product $\mathbb{R}^d \times \tilde{B}$. Thus the dimension of the Euclidean factor of $\tilde{M}$ is at least d. In other words, $d \leq \dim(V)$, and from Corollary 4 b) we infer that the holonomy group of M is compact.

Finally we prove e). By assumption the Ricci tensor of $\tilde{M}$ is parallel. Thus each factor of the de Rham decomposition $\tilde{M} = \tilde{N}_1 \times \cdots \times \tilde{N}_k \times \mathbb{R}^l$ is Einstein. We assume that the Einstein constant of $\tilde{N}_j$ is nonnegative if and only if $j \leq h$. By Myers theorem the factors with positive Einstein constants are compact, and we have seen in section 3 that the Ricci-flat factors are compact, too. Hence, $\tilde{N}_1 \times \cdots \times \tilde{N}_h$ is compact. Notice that the Ricci curvature of $\tilde{M}' := \tilde{N}_{h+1} \times \cdots \tilde{N}_k \times \mathbb{R}^l$ is nonpositive. Furthermore, for a vector $v \in T\tilde{M}'$ we have $\mathrm{Ric}(v, v) = 0$ if and only if v is tangential to the Euclidean factor.

Let Π' denote the image of the deck transformation group Π under the projection $\mathrm{pr} \colon \mathrm{Iso}(\tilde{M}) \to \mathrm{Iso}(\tilde{M}')$. Observe that $\mathsf{L}'_{\mathbb{R}} := \mathrm{pr}(\mathsf{L}_{\mathbb{R}}) \cong \mathbb{R}^d$ is a closed subgroup that is normalized by Π'. Moreover, the action of $\mathsf{L}'_{\mathbb{R}}$ is free and the corresponding orbits are closed. Thus

$$\tilde{\sigma}' \colon \tilde{M}' \to \tilde{B}' := \tilde{M}'/\mathsf{L}'_{\mathbb{R}}, \ \ q \mapsto \mathsf{L}'_{\mathbb{R}} \star q$$

is a Riemannian submersion. We define a function $f' \colon \tilde{M}' \to \mathbb{R}$ analogously to the proof of Corollary 4 c). Since the Ricci curvature of $\tilde{M}'$ is nonpositive, we can use the same argument in order to show that f' is subharmonic and accordingly constant. Now we infer from the equality case in inequality (5) that $\mathrm{Ric}(v, v) = 0$ for any vertical vector $v \in T\tilde{M}'$. So the vertical distribution of $\tilde{\sigma}'$ is everywhere tangential to the Euclidean factor. In particular, $\dim(V) = l \geq \dim(\mathsf{L}'_{\mathbb{R}}) = d$, and thus the holonomy group of M is compact by Corollary 4 b).

6.2 The Proof of Corollary 5

First we prove part b) of the corollary. By Corollary 4 a finite cover $\hat{M}$ of M is a torus bundle over a compact manifold B. Moreover, the dimension of B is at most $\dim(M) - 4 = 2$. It is clear from Corollary 3 that $\pi_1(M)$ is not abelian up to finite index, and thus $\pi_1(B)$ is necessarily infinite. By the classification of compact 2-manifolds this implies that B is aspherical. Consequently, the universal covering space of M is diffeomorphic to $\mathbb{R}^6$.

It remains to prove a). The inequality $\dim(M) \geq 5$ is an immediate consequence of Corollary 4 and part b) of the addendum. So we only have to consider the case $\dim(M) = 5$. Let $\tilde{M}$ be the universal covering

of M, and let $\Pi \subset \mathrm{Iso}(\tilde{M})$ be the group of covering transformations. From the proof of Theorem 2 and from Corollary 3 we deduce that there is

(1) a free abelian normal subgroup $\mathsf{L} \subset \Pi$ of rank at least four and

(2) a connected, simply connected abelian subgroup $\mathsf{L}_{\mathbb{R}} \subset \mathrm{Iso}(\tilde{M})$ normalized by Π and containing L as a discrete, cocompact subgroup.

Moreover, we have seen in the proof of Corollary 4 that $\mathsf{L}_{\mathbb{R}}$ acts freely on $\tilde{M}$ and that the corresponding orbits are closed. As $\tilde{M}$ is not flat, the group $\mathsf{L}_{\mathbb{R}}$ does not act transitively. Therefore $\mathsf{L}_{\mathbb{R}} \cong \mathbb{R}^4$ and the complete, simply connected manifold $\tilde{M}/\mathsf{L}_{\mathbb{R}}$ is isometric to $\mathbb{R}$. Let

$$\tilde{\sigma} \colon \tilde{M} \to \mathbb{R}$$

be the corresponding Riemannian submersion and choose a unit vector $v \in T\tilde{M}$ that is horizontal with respect to $\tilde{\sigma}$. Notice that the map

$$f \colon \mathsf{L}_{\mathbb{R}} \times \mathbb{R} \to \tilde{M}, \ \ (g, t) \mapsto g \star \exp(tv)$$

is a diffeomorphism. We identify $\mathsf{L}_{\mathbb{R}} \times \mathbb{R}$ with $\tilde{M}$ via f. The action of $\mathsf{L}_{\mathbb{R}}$ on $\mathsf{L}_{\mathbb{R}} \times \mathbb{R} = \tilde{M}$ is given by left translation on the first factor. Moreover, the lines $t \mapsto (g, t)$ are horizontal geodesics with respect to $\tilde{\sigma}$.

Consider the closed subgroup $\mathsf{P} := \mathsf{L}_{\mathbb{R}} \cdot \Pi$, the line $\{e\} \times \mathbb{R} = \mathbb{R}$ and the subgroup $\mathsf{H} := \{g \in \mathsf{P} \mid g \star \mathbb{R} = \mathbb{R}\}$. Since the action of $\mathsf{L}_{\mathbb{R}}$ on the horizontal lines is simply transitive, it follows that H is a group complement of $\mathsf{L}_{\mathbb{R}}$ in P, i.e., $\mathsf{H} \cap \mathsf{L}_{\mathbb{R}} = \{e\}$ and $\mathsf{P} = \mathsf{L}_{\mathbb{R}} \cdot \mathsf{H}$. Hence P is isomorphic to a semidirect product $\mathsf{L}_{\mathbb{R}} \rtimes_\beta \mathsf{H}$ where the homomorphism $\beta \colon \mathsf{H} \to \mathrm{Aut}(\mathsf{L}_{\mathbb{R}}) \cong \mathsf{GL}(4, \mathbb{R})$ is induced by conjugation. We identify $\mathsf{L}_{\mathbb{R}} \rtimes_\beta \mathsf{H}$ with P via the isomorphism $\mathsf{L}_{\mathbb{R}} \rtimes_\beta \mathsf{H} \to \mathsf{P}, \ (b, h) \mapsto bh$.

By construction H acts on $\{e\} \times \mathbb{R} = \mathbb{R}$. In particular, $h \star t$ is defined for $h \in \mathsf{H}$ and $t \in \mathbb{R}$. Observe that the action of $\mathsf{P} = \mathsf{L}_{\mathbb{R}} \rtimes_\beta \mathsf{H}$ on $\mathsf{L}_{\mathbb{R}} \times \mathbb{R}$ is given by

$$(b, h) \star (v, t) \ = \ \bigl(b \cdot \beta(h)(v), h \star t\bigr)$$

for $(b, h) \in \mathsf{L}_{\mathbb{R}} \rtimes_\beta \mathsf{H}$ and $(v, t) \in \mathsf{L}_{\mathbb{R}} \times \mathbb{R} = \tilde{M}$.

The group H acts isometrically, discontinuously and cocompactly on $\mathbb{R}$. Thus H contains an infinite cyclic normal subgroup of finite index. From Corollary 3 we infer that the image of the homomorphism $\beta \colon \mathsf{H} \to \mathrm{Aut}(\mathsf{L}) \subset \mathrm{Aut}(\mathsf{L}_{\mathbb{R}})$ is infinite. Let Z be the Zarisky closure of

$\beta(\mathsf{H})$ in $\mathrm{Aut}(\mathsf{L}_{\mathbb{R}}) \cong \mathsf{GL}(4, \mathbb{R})$; see [8, p. 7-12] for the definition and properties of the Zarisky closure. Then Z has only finitely many connected components, and the identity component is abelian. It is easy to see that there is a simply connected subgroup Z' with finitely many connected components such that $\beta(\mathsf{H})$ is a discrete, cocompact subgroup of Z'.

Choose $h_0 \in \mathsf{H}$ such that $\beta(h_0)$ generates an infinite cyclic subgroup of $\beta(\mathsf{H}) \cap \mathsf{Z}_0'$ where Z_0' is the identity component of Z'. We can assume that the group generated by h_0 is a normal subgroup of H because we replace h_0 by a power of h_0. The exponential map of the simply connected abelian Lie group Z_0' is a diffeomorphism, and hence there is a unique vector $X \in \mathfrak{z}'$ in the Lie algebra of Z' for which $\exp(X) = \beta(h_0)$. Since the action of H on $\mathbb{R}$ is isometric, there is a number $t_0 \in \mathbb{R}$ satisfying $h_0 \star t = t + t_0$ for all $t \in \mathbb{R}$. It is sufficient to consider the case $t_0 > 0$ because we can otherwise pass from h_0 to h_0^{-1}. After scaling we have $t_0 = 1$. Now we turn $\mathsf{L}_{\mathbb{R}} \times \mathbb{R}$ into a solvable Lie group S by setting

$$(u, s) \cdot (v, t) := \big(u \cdot \exp(sX)(v), s + t\big)$$

for all $(u, s), (v, t) \in \mathsf{L}_{\mathbb{R}} \times \mathbb{R} = \mathsf{S}$. With respect to this group structure P acts on S by affine diffeomorphisms, and the subgroup of P generated by h_0 and $\mathsf{L}_{\mathbb{R}}$ acts on S by left translation. Consequently, there is a finite group $\mathsf{F} \subset \mathrm{Aut}(\mathsf{S})$ such that P can be viewed as a subgroup of $\mathsf{S} \rtimes \mathsf{F}$. Since Π is a discrete, cocompact subgroup of P, it becomes a discrete, cocompact subgroup of $\mathsf{S} \rtimes \mathsf{F}$ and the manifold M is diffeomorphic to S/Π. The group Π is torsion free because M is aspherical.

It remains to prove that M also admits a local homogeneous metric with a noncompact holonomy group: According to Theorem 2 the real representation $\tilde{\rho}: \Pi \to \mathsf{GL}(\mathsf{L}) \subset \mathsf{GL}(\mathsf{L}_{\mathbb{R}})$ induced by conjugation decomposes as a direct sum $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$, where $\tilde{\rho}_1$ is equivalent to the holonomy representation ρ_1 of $\Pi = \pi_1(M)$. Let $\mathsf{L}_{\mathbb{R}} = \mathsf{U} \oplus \mathsf{V}$ be the corresponding decomposition of the vector group $\mathsf{L}_{\mathbb{R}}$. Then U and V are normal subgroups of $\mathsf{S} \rtimes \mathsf{F}$. Furthermore, the representation of $\mathsf{S} \rtimes \mathsf{F}$ in U induced by conjugation is orthogonal with respect to a suitable scalar product. It is straightforward to check that $\mathsf{S} \rtimes \mathsf{F}$ is isomorphic to a semidirect product $\mathsf{U} \times_{\beta_1} ((\mathsf{S}/\mathsf{U}) \rtimes \mathsf{F})$, where β_1 is an orthogonal representation. Analogously to section 2 one can now construct a local homogeneous metric with a noncompact holonomy group on $M \cong \mathsf{S}/\Pi$.

7. Construction of manifolds with prescribed holonomy representations

7.1 Finite index embeddings into semidirect products

If H and G are groups, and $\beta\colon H \to \mathrm{Aut}(G)$ is a homomorphism, then we will subsequently always identify the groups H and G with the subgroups $G \times \{e\}$ and $\{e\} \times H$ of the corresponding semidirect product $G \rtimes_\beta H$. This subsection is devoted to the proof of the following lemma which is needed for the proof of Theorem 6.

Lemma 7.1. *Let Π be a group, L a finitely generated, free abelian normal subgroup, and let $H \subset \Pi$ be a subgroup such that $H \cap L = \{e\}$ and $H \cdot L$ has finite index in Π. Then there is a semidirect product $\mathbb{Z}^d \rtimes_\beta (\Pi/L)$ and a monomorphism $\iota\colon \Pi \hookrightarrow \mathbb{Z}^d \rtimes_\beta (\Pi/L)$ satisfying the following two conditions.*

(1) $\iota(\Pi)$ has finite index in $\mathbb{Z}^d \rtimes_\beta (\Pi/L)$.

(2) $\mathrm{pr}(\iota(g)) = g \cdot L$ for $g \in \Pi$, where $\mathrm{pr}\colon \mathbb{Z}^d \rtimes_\beta (\Pi/L) \to \Pi/L$ denotes the natural projection.

Proof. We may assume that $H \cdot L$ is a normal subgroup of Π because we can replace H by a subgroup of finite index. Fix an isomorphism $i\colon \mathbb{Z}^d \to L$ where $d = \mathrm{rank}(L)$. The conjugate action of Π on L induces then a representation $\alpha\colon \Pi \to \mathsf{GL}(\mathbb{Z}^d) \subset \mathsf{GL}(\mathbb{Q}^d)$. Consider the semidirect product $\mathbb{Q}^d \rtimes_\alpha \Pi$ and the normal subgroup

$$N := \big\{ (-v, i(v)) \mid v \in \mathbb{Z}^d \big\}.$$

The projection $\pi\colon \mathbb{Q}^d \rtimes_\alpha \Pi \to G := (\mathbb{Q}^d \rtimes_\alpha \Pi)/N$ maps Π and $\mathbb{Q}^d$ monomorphically onto subgroups of G. Moreover, $G/\pi(\mathbb{Q}^d) \cong \Pi/L$, and therefore we get an exact sequence

$$\{0\} \to \mathbb{Q}^d \xrightarrow{\pi} G \xrightarrow{\bar{q}} \Pi/L \to \{1\},$$

where $\bar{q}$ is characterized by $\bar{q} \circ \pi(0, g) = g \cdot L$. The natural projection $\Pi \to \Pi/L$ maps H injectively onto a subgroup of Π/L. Thus we can view H as a normal subgroup of finite index in Π/L. Notice that with respect to this identification $\bar{q} \circ \pi_{|H} = \mathrm{id}$.

Choose representatives $c_1, \ldots, c_q \in \Pi$ of the factor group $\Pi/(L \cdot H)$ where $q = \mathrm{ord}\big(\Pi/(H \cdot L)\big)$. Then the elements $b_i := c_i \cdot L$ $(i = 1, \ldots, q)$

are representatives of the factor group $(\Pi/\mathsf{L})/\mathsf{H}$, and we can define a set-theoretical section

$$s\colon \Pi/\mathsf{L} \to \mathsf{G} \quad \text{by} \quad s(h \cdot b_i) := \pi(h) \cdot c_i \quad \text{for } h \in \mathsf{H},\, i = 1, \ldots, q.$$

Observe that for $f \in \Pi/\mathsf{L}$ the map $s_f(a) := s(f)^{-1} s(fa)$ is a section, too. If $f_1 \cdot \mathsf{H} = f_2 \cdot \mathsf{H}$, then $s_{f_1} = s_{f_2}$; accordingly s_f is well-defined for $f \in (\Pi/\mathsf{L})/\mathsf{H}$. It is elementary to verify that the section $h\colon \Pi/\mathsf{L} \to \mathsf{G}$ given by

$$h(a) := s(a) \cdot \pi \left(\tfrac{1}{q} \sum_{f \in (\Pi/\mathsf{L})/\mathsf{H}} \pi_{|\mathbb{Q}^d}^{-1} \left(s(a)^{-1} s_f(a) \right) \right)$$

is a homomorphism. Notice that $h_{|\mathsf{H}} = \pi_{|\mathsf{H}}$. Put $\mathsf{G}_k := \pi \left((\tfrac{1}{k!}\mathbb{Z}^d) \rtimes_\alpha \Pi \right)$ for a positive integer k. Clearly, $\bigcup_{k \geq 1} \mathsf{G}_k = \mathsf{G}$. Since the subgroup $h(\mathsf{H}) \subset \mathsf{G}_1$ has finite index in $h(\Pi/\mathsf{L})$, it follows that $h(\Pi)$ is contained in G_k for a sufficiently large number k. Observe that the two groups $h(\Pi/\mathsf{L})$ and $\pi(\tfrac{1}{k!}\mathbb{Z}^d)$ have trivial intersection and that their product equals G_k. Therefore G_k is canonically isomorphic to a semidirect product $\mathbb{Z}^d \rtimes_\beta \Pi/\mathsf{L}$. Moreover, the embedding $\iota := \pi_{|\Pi}\colon \Pi \to \mathsf{G}_k \cong \mathbb{Z}^d \rtimes_\beta \Pi/\mathsf{L}$ satisfies the two conditions of the lemma.   q.e.d.

7.2   The proof of Theorem 6

In view of Lemma 4.6 we may assume that $\mathbb{Z}\text{-rk}(\tilde{\rho}_1) = \text{rank}(\mathsf{L})$.

Choose an embedding $\iota\colon \Pi \hookrightarrow \mathbb{Z}^d \rtimes_\beta \Pi/\mathsf{L}$ as described in Lemma 7.1 and identify Π with its image. Moreover, we view $\mathbb{Z}^d \rtimes_\beta \Pi/\mathsf{L}$ as a subgroup of $\mathbb{R}^d \rtimes_\beta \Pi/\mathsf{L}$. By construction $\beta \circ \text{pr}_{|\Pi}$ is equivalent to $\tilde{\rho}\colon \Pi \to \mathsf{GL}(\mathsf{L} \otimes_{\mathbb{Z}} \mathbb{R})$, where $\text{pr}\colon \mathbb{R}^d \rtimes_\beta \Pi/\mathsf{L} \to \Pi/\mathsf{L}$ denotes the natural projection. Consequently, there is a decomposition $\beta = \beta_1 \oplus \beta_2$ such that $\beta_i \circ \text{pr}_{|\Pi}$ is equivalent to $\tilde{\rho}_i$, $i = 1, 2$. Let $\mathbb{R}^d = U_1 \oplus U_2$ be the corresponding decomposition of $\mathbb{R}^d$.

Since Π is finitely presented and L is finitely generated, the factor group Π/L is finitely presented, too. Recall that a finitely generated and finitely presented group can be realized as the fundamental group of a compact 4-manifold. Thus Π/L acts isometrically, freely, discontinuously and cocompactly on a connected, simply connected, complete Riemannian 4-manifold (N, g_0). By disturbing the metric g_0 if necessary, we can assume that there is a point $p_0 \in N$ and a number $r > 0$ such that the ball $(B_r(p_0), g_0)$ has negative Ricci curvature. The group

$U_2 \rtimes_{\beta_2} (\Pi/\mathsf{L})$ acts on $N \times U_2$ by

$$(w, f) \star (p, \tilde{w}) := \big(f \star p, \beta_2(f)(\tilde{w}) + w\big)$$

for $(w, f) \in U_2 \rtimes_{\beta_2} (\Pi/\mathsf{L})$ and $(p, \tilde{w}) \in N \times U_2$. It is easy to see that there is a metric g on $N \times U_2$ such that the following holds:

(1) The action of $U_2 \times_{\beta_2} (\Pi/\mathsf{L})$ on $(N \times U_2, g)$ is isometric.

(2) For any nontrivial Killing field Y that is induced by the action of U_2 the norm $\|Y\|$ is not constant.

(3) There is a positive number $r_1 \leq r$ such that $B_{r_1}(p_0) \times U_2 \subset N \times U_2$ carries the product metric $g_0 \times \langle \cdot, \cdot \rangle_2$, where $\langle \cdot, \cdot \rangle_2$ is a scalar product on U_2.

We want to convince ourselves that the Euclidean factor of $(N \times U_2, g)$ is trivial, otherwise there would be a nontrivial parallel Killing field Y on $(N \times U_2, g)$. Since $B_{r_1}(p) \times U_2$ carries a product metric, and $(B_{r_1}(p), g_0)$ has negative Ricci curvature, it follows that the restriction of Y to $B_{r_1}(p) \times U_2$ is a Killing field induced by the action of U_2. But $B_{r_1}(p) \times U_2$ is open in $N \times U_2$, and hence Y itself is induced by the action of U_2. By (2) this implies that the norm of Y is not constant – a contradiction.

By the assumption (ii) of the theorem there is a scalar product $\langle \cdot, \cdot \rangle$ on U_1 with respect to which β_1 becomes an orthogonal representation. Consider the Riemannian product

$$\tilde{M} := \big((N \times U_2, g)\big) \times \big(U_1, \langle \cdot, \cdot \rangle\big).$$

The group $\mathbb{R}^l \rtimes_\beta \Pi/\mathsf{L} \cong (U_1 \times U_2) \rtimes_{(\beta_1, \beta_2)} \Pi/\mathsf{L}$ acts freely, isometrically and cocompactly on $\tilde{M}$ by

$$(v, w, f) \star ((p, \tilde{w}), \tilde{v}) := \big((f, w) \star (p, \tilde{w}), \beta_1(f)(\tilde{v}) + v\big)$$

for $(v, w, f) \in (U_1 \times U_2) \rtimes_{(\beta_1, \beta_2)} \Pi/\mathsf{L}$ and $((p, \tilde{w}), \tilde{v}) \in \tilde{M}$. Moreover, the orbits of this action are closed. Since Π is a discrete, cocompact subgroup of $\mathbb{R}^l \rtimes_\beta \Pi/\mathsf{L}$, we obtain a discontinuous, cocompact, free action of Π on $\tilde{M}$. The holonomy representation of $\Pi = \pi_1(\tilde{M}/\Pi)$ is then equivalent to $\beta_1 \circ \mathrm{pr}_{|\Pi} \cong \tilde{\rho}_1$.

Notice that the dimension of the constructed manifold is $\mathbb{Z}\text{-rk}(\tilde{\rho}_1) + 4$, as claimed in the introduction.

7.3 The proof of Corollary 7

By Theorem 2 (iv) the integer rank of a holonomy representation is finite and hence the condition is necessary.

Suppose now conversely that there is a finitely generated group Λ which is invariant under Φ. For the finitely generated group Φ we can find a free group H of finite rank and an epimorphism $\psi \colon \mathsf{H} \to \Phi$.

We let $\tilde{\psi}$ denote the induced representation in the free $\mathbb{Z}$-module Λ and put $\Pi := \Lambda \rtimes_{\tilde{\psi}} \mathsf{H}$. Let $\rho \colon \Pi \to \mathsf{GL}(\Lambda)$ be the representation that is induced by conjugation. Notice that this representation coincides with $\tilde{\psi} \circ \mathrm{pr}$ where $\mathrm{pr} \colon \Lambda \rtimes_{\tilde{\psi}} \mathsf{H} \to \mathsf{H}$ denotes the natural projection. If we regard ρ as a rational representation in $\mathrm{span}_{\mathbb{Q}}(\Lambda) \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, it is completely reducible; see Lemma 4.1. By Lemma 4.3 the corresponding real representation $\tilde{\rho} \colon \Pi \to \mathsf{GL}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ is completely reducible, too.

The natural inclusion $\Lambda \hookrightarrow \mathbb{R}^l$ induces an epimorphism $f \colon \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^l$. Clearly, f is an equivariant homomorphism from $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}, \tilde{\rho})$ to $(\mathbb{R}^l, \psi \circ \mathrm{pr})$. Since $\tilde{\rho}$ is fully reducible, it decomposes as a direct sum $\tilde{\rho} = \tilde{\rho}_1 \oplus \tilde{\rho}_2$, where $\tilde{\rho}_1$ is equivalent to the orthogonal representation $\psi \circ \mathrm{pr}$. Finally, Π is finitely presented, and Theorem 6 implies that $\tilde{\rho}_1 \cong \psi \circ \mathrm{pr}$ is a holonomy representation. This completes the proof of the corollary, because Φ is the image of $\psi \circ \mathrm{pr}$.

7.4 Chevalley bases

This subsection provides some basic facts on Chevalley bases which are needed in the next subsection. First we want to recall the concept of a Chevalley basis of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let B be the Killing form on $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and let $\Phi \subset \mathfrak{h}^*$ be the corresponding root system. Consider the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}^{\alpha}.$$

For each $\alpha \in \Phi$ we let $h_{\alpha} \in \mathfrak{h}$ denote the corresponding coroot, that is the element with $\alpha(h_{\alpha}) = 1$ and $B(h_{\alpha}, X) = \alpha(X) \cdot B(h_{\alpha}, h_{\alpha})$ for all $X \in \mathfrak{h}$. Furthermore, we fix a lexicographic ordering on $\mathrm{span}_{\mathbb{R}}(\Phi)$, and let $\Delta \subset \Phi$ denote the positive simple roots. Finally we choose a nonzero vector $e_{\alpha} \in \mathfrak{g}^{\alpha}$, $\alpha \in \Phi$. The vectors in

$$E := \{ h_{\alpha} \mid \alpha \in \Delta \} \cup \{ e_{\alpha} \mid \alpha \in \Phi \}$$

are called a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$ if and only if the following hold

$$[e_\alpha, e_{-\alpha}] = h_\alpha,$$
$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$$

with complex numbers $c_{\alpha,\beta}$ satisfying the relation $c_{-\alpha,-\beta} = -c_{\alpha,\beta}$. As a consequence the structure constants $c_{\alpha,\beta}$ are determined up to the sign:

$$c_{\alpha,\beta} = 0, \text{ if } \alpha + \beta \notin \Phi,$$
$$c_{\alpha,\beta} = \pm(r+1), \text{ if } \alpha + \beta \in \Phi \text{ and } r = \max\{r \in \mathbb{Z} \mid \beta - r\alpha \in \Phi\}.$$

Recall that a Chevalley basis always exists; see [6, p. 144].

Lemma 7.2. *Let B and E be two Chevalley bases of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then there is an inner automorphism τ of $\mathfrak{g}_{\mathbb{C}}$ with $\tau(B \cup -B) = E \cup -E$.*

Proof. Recall that for any Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}_{\mathbb{C}}$ there is an inner automorphism τ_1 of $\mathfrak{g}_{\mathbb{C}}$ with $\tau_1(\tilde{\mathfrak{h}}) = \mathfrak{h}$. Moreover, the Weyl group corresponding to $\mathfrak{h}$ acts transitively on the positive systems. In other words, without loss of generality the Chevalley basis B also corresponds to $\mathfrak{h}$ and Δ. Let $\{b_\alpha\} := B \cap \mathfrak{g}_{\mathbb{C}}^\alpha$ for $\alpha \in \Phi$. Since the roots in Δ are linearly independent, there is an element $v \in \mathfrak{h}$ with $e_\alpha = \exp(\mathrm{ad}_v) b_\alpha$ for all $\alpha \in \Delta$.

So we can in addition assume $e_\alpha = b_\alpha$ for all $\alpha \in \Delta$. Notice that the structure constants corresponding to the two bases coincide up to signs. Therefore $\pm e_\alpha = \pm b_\alpha$ for any positive root α. Finally, for any root α we have $[e_\alpha, e_{-\alpha}] = [b_\alpha, b_{-\alpha}] = h_\alpha$ and thus $\pm e^\alpha = \pm b^\alpha$ for any root. But this proves $B \cup -B = E \cup -E$.   q.e.d.

Lemma 7.3. *Let $\mathfrak{k}$ be a real semisimple Lie algebra of compact type, and let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{k}_{\mathbb{C}}$ be its complexification. Then there is a Chevalley basis E of $\mathfrak{k}_{\mathbb{C}}$ for which the following hold:*

a) *$c_{\mathfrak{k}}(E \cup -E) = E \cup -E$, where $c_{\mathfrak{k}}$ denotes the conjugation with respect to $\mathfrak{k}$.*

b) *There is a generator system $v_1, \ldots, v_h \in \langle E \rangle_{\mathbb{Z}[i]} := \mathrm{span}_{\mathbb{Z}[i]}(E)$ of $\mathfrak{k}$ such that $\exp(\mathbb{R} v_i)$ is compact, $i = 1, \ldots, h$, where $\exp \colon \mathfrak{k} \to \mathsf{K}$ denotes the exponential mapping from $\mathfrak{k}$ onto the corresponding simply connected compact Lie group.*

c) *The natural projection from the automorphism group of $\mathfrak{k}$ onto the group of outer automorphism,*

$$\mathrm{Aut}(\mathfrak{k}) \to \mathrm{Out}(\mathfrak{k}) := \mathrm{Aut}(\mathfrak{k}) / \mathrm{Int}(\mathfrak{k}),$$

restricts to an epimorphism

$$\mathrm{pr}_{|\mathsf{F}} \colon \mathsf{F} \to \mathrm{Out}(\mathfrak{k}),$$

where

$$\mathsf{F} \subset \mathrm{Aut}(\mathfrak{k}) \subset \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$$

is the finite group

$$\mathsf{F} := \{\sigma \in \mathrm{Aut}(\mathfrak{k}) \mid \sigma(E \cup -E) = E \cup -E\}.$$

Proof. Let $\Delta \subset \Phi$ and

$$E = \{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$$

be as above. Define $\mathfrak{k}'$ as the $\mathbb{R}$-span of the following vectors

$$ih_\alpha,\, i(e_\alpha + e_{-\alpha}),\, e_\alpha - e_{-\alpha}, \quad \alpha \in \Phi.$$

It is straightforward to check that $\mathfrak{k}'$ is a compact real form of $\mathfrak{k}_{\mathbb{C}}$, and obviously $E \cup -E$ is invariant under the conjugation $c_{\mathfrak{k}'}$ corresponding to $\mathfrak{k}'$. Since the compact real forms $\mathfrak{k}$ and $\mathfrak{k}'$ are isomorphic via an inner automorphism of $\mathfrak{k}_{\mathbb{C}}$, the statement a) of the lemma follows. Moreover, in the following we may assume that $\mathfrak{k}' = \mathfrak{k}$.

Notice that for $\alpha \in \Phi$ the elements $ih_\alpha, i(e_\alpha + e_{-\alpha})$, and $(e_\alpha - e_{-\alpha})$ generate a Lie algebra isomorphic to $\mathfrak{su}(2, \mathbb{R})$. In particular, $\exp(\mathbb{R}{\cdot}ih_\alpha)$, $\exp\big(\mathbb{R} \cdot i(e_\alpha + e_{-\alpha})\big)$ and $\exp\big(\mathbb{R} \cdot (e_\alpha - e_{-\alpha})\big)$ are compact groups, and hence we have found a generator system of $\mathfrak{k}$ satisfying b).

It remains to prove c). So let $\sigma \in \mathrm{Aut}(\mathfrak{k})$. Notice that $\sigma(E)$ is again a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$. By Lemma 7.2 there is an inner automorphism $\tau \in \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ such that $\tau(\sigma(E \cup -E)) = E \cup -E$. Furthermore, in view of the proof of Lemma 7.2 we may assume that $\tau \circ \sigma$ leaves the sets $\{h_\alpha \mid \alpha \in \Delta\}$ and $\{e_\alpha \mid \alpha \in \Delta\}$ invariant.

Using $[e_\alpha, e_{-\alpha}] = h_\alpha$ we see that $\tau \circ \sigma$ leaves the set $\{e_{-\alpha} \mid \alpha \in \Delta\}$ invariant as well. Consequently, the set

$$\{ih_\alpha \mid \alpha \in \Delta\} \cup \{i(e_\alpha + e_{-\alpha}) \mid \alpha \in \Delta\} \cup \{e_\alpha - e_{-\alpha} \mid \alpha \in \Delta\} \subset \mathfrak{k}$$

is invariant under $\tau \circ \sigma$. Clearly, $\mathfrak{k}$ is the minimal real subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing these elements. Thus $\tau(\mathfrak{k}) = \tau \circ \sigma(\mathfrak{k}) = \mathfrak{k}$. But this implies $\tau \in \mathrm{Int}(\mathfrak{k})$. So $\tau \circ \sigma \in \mathsf{F}$ and $\mathrm{pr}(\sigma) = \mathrm{pr}(\tau \circ \sigma)$.   q.e.d.

7.5 Three lemmas

Two of the three lemmas we prove in this subsection are special cases of Corollary 8. In the next subsection we will use these lemmas to establish Corollary 8 in full generality.

Lemma 7.4. *Let* $\mathsf{K} \subset \mathsf{GL}(l, \mathbb{C})$ *be a connected compact semisimple Lie group, and let* $\mathfrak{k} \subset \mathsf{GL}(l, \mathbb{C})$ *be its Lie algebra. There is a finitely generated dense subgroup* $\Phi_1 \subset \mathsf{K}$ *for which the following hold:*

a) *A finitely generated cocompact subgroup* $\Lambda_1 \subset \mathbb{C}^l$ *is invariant under* Φ_1.

b) *The group* Φ_1 *is normalized by*

$$\mathsf{B} := \{A \in \mathsf{GL}(l, \mathbb{C}) \mid \mathrm{Ad}_{A|\mathfrak{k}} \in \mathsf{F}\},$$

where $\mathsf{F} \subset \mathrm{Aut}(\mathfrak{k})$ *is the finite group from Lemma 7.3 c).*

Sublemma. *For each positive integer* m *there is an element* $z \in \mathsf{S}^1 \subset \mathbb{C}$ *of infinite order such that* $\frac{z-1}{m}$ *is an algebraic integer.*

Proof. Recall that the ring of real algebraic integers is dense in $\mathbb{R}$. Thus we can find an algebraic integer $a \in \mathbb{R}$ satisfying $0 < a < \frac{1}{m^2}$. Moreover, we may assume that the field extension $\mathbb{Q} \subset \mathbb{Q}[a]$ is not normal. Then the element

$$z := 1 - m^2 a + i\big(2m^2 a - m^4 a^2\big)^{1/2} \in \mathsf{S}^1 \subset \mathbb{C}$$

is not a root of unity. Since $\frac{z-1}{m} = -ma + i\big(2a - ma^2\big)^{1/2}$ is an algebraic integer, we are done.   q.e.d.

Proof of Lemma 7.4. Let $\mathfrak{k} \subset M(l, \mathbb{C})$ denote the Lie algebra of K and $\mathfrak{k}_\mathbb{C} \subset M(l, \mathbb{C})$ its complexification. Choose a Chevalley basis E of $\mathfrak{k}_\mathbb{C}$ as stated in Lemma 7.3. From [6, p. 156] it follows that there is a complex lattice $L \subset \mathbb{C}^l$ which is invariant under the matrices in E; by a complex lattice we mean that L is a free abelian subgroup of $\mathbb{C}^l$ with $\mathrm{rank}(L) = l$ and $\mathrm{span}_\mathbb{C}(L) = \mathbb{C}^l$. Without loss of generality we may assume that $L = \mathbb{Z}^l$ and accordingly

$$E \subset M(l, \mathbb{Z}),$$

because we can replace K by a conjugate subgroup. Choose a generator system $v_1, \ldots, v_h \in \mathfrak{k} \cap \langle E \rangle_{\mathbb{Z}[i]}$ of $\mathfrak{k}$ as stated in Lemma 7.3 b).

Furthermore we may assume that the set $\{v_1, \ldots, v_h\}$ is invariant under the natural action of F, where $\mathsf{F} \subset \mathrm{Aut}(\mathfrak{k})$ is the finite group from Lemma 7.3 c).

Since $\exp(\mathbb{R}v_h)$ is compact, there is a positive real number λ_i such that the Eigenvalues of $\lambda_i v_i$ are contained in $\mathbb{Z} \cdot i$. Because of $v_i \in M(l, \mathbb{Z}[i])$ the number λ_i is algebraic. Set $\mathbb{K} := \mathbb{Q}[i, \lambda_1, \ldots, \lambda_h]$, and choose a matrix $A_i \in M(l, \mathbb{K})$ such that $A_i \cdot v_i \cdot A_i^{-1}$ is diagonal. Evidently, we can find a positive integer j such that the coefficients of the matrices $j \cdot A_i$ and $j \cdot A_i^{-1}$ are algebraic integers for all i. For $m = j^2$ we choose an algebraic integer $z \in \mathsf{S}^1 \subset \mathbb{C}$ as stated in the sublemma. Define $\Phi_1 \subset \mathsf{K}$ as the group generated by the following set

$$Gen := \left\{ B \in \mathrm{Exp}(\mathbb{R}v_i) \ \middle| \ \begin{array}{c} \text{the Eigenvalues of } B \text{ lie in} \\ \{z^k \mid k \in \mathbb{Z}\}, i = 1, \ldots, h \end{array} \right\}.$$

Notice that $A_i B A_i^{-1}$ is a diagonal matrix for $B \in \mathrm{Exp}(\mathbb{R}v_i) \cap Gen$. By definition of z and j the coefficients of the matrices

$$\tfrac{1}{j^2}\left(A_i B A_i^{-1} - I\right), \ \ j \cdot A_i \text{ and } j \cdot A_i^{-1}$$

are algebraic integers. Therefore the matrix B consists of algebraic integers as well. Let $\widehat{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[i, \lambda_1, \ldots, \lambda_k, z]$. We have proved $\Phi_1 \subset \mathsf{GL}(l, \widehat{\mathbb{Z}})$, and thereby Φ_1 leaves the finitely generated cocompact subgroup $\widehat{\mathbb{Z}}^l \subset \mathbb{C}^l$ invariant. Notice that $Gen \cap \mathrm{Exp}(\mathbb{R}v_i)$ is a dense subgroup of $\mathrm{Exp}(\mathbb{R}v_i)$. In particular, the vectors $v_1, \ldots, v_h$ are contained in the Lie algebra of the closure of Φ_1. Hence Φ_1 is dense in the connected Lie group K.

As $\{v_1, \ldots, v_h\}$ is under invariant $\mathsf{F} \subset \mathrm{Aut}(\mathfrak{k})$, the set Gen is invariant under conjugation by elements in the group B. Thus B normalizes Φ_1.   q.e.d.

Lemma 7.5. *Let K be a compact Lie group, and let K_0 be the identity component of K. Then there is a finite group $\mathsf{E} \subset \mathsf{K}$ that intersects each connected component of K.*

Proof. Choose a maximal toral subgroup $\mathsf{T} \subset \mathsf{K}_0$, and let H denote the normalizer of T. Clearly, T is the identity component of H, and H intersects each connected component of K. Hence it is sufficient to prove the statement for H instead of K.

In other words, without loss of generality we may assume that K_0 is a torus. Set $k := \mathrm{ord}(\pi_0(\mathsf{K}))$ and $\mathsf{L} := \{a \in \mathsf{K}_0 \mid a^k = e\}$. Notice that L is a finite characteristic subgroup of K. Let $\mathrm{pr} \colon \mathsf{K} \to \mathsf{K}/\mathsf{L}$ denote the

projection. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
\{1\} & \to & \mathsf{K}_0 & \overset{\iota}{\to} & \mathsf{K} & \overset{q}{\to} & \pi_0(\mathsf{K}) & \to & \{1\} \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \to & \mathsf{K}_0/\mathsf{L} & \overset{\iota'}{\to} & \mathsf{K}/\mathsf{L} & \overset{q'}{\to} & \pi_0(\mathsf{K}) & \to & \{1\},
\end{array}
$$

where the horizontal sequences are exact, and the vertical maps are the natural projections. Similarly to subsection 7.1 we can employ a set-theoretical section $s\colon \pi_0(\mathsf{K}) \to \mathsf{K}$ to define a homomorphism $h\colon \pi_0(\mathsf{K}) \to \mathsf{K}/\mathsf{L}$ as follows:

$$
h(a) := \operatorname{pr}(s(a)) \cdot p_k\bigg(\prod_{f \in \pi_0(\mathsf{K})} \big(s(a)^{-1} s_f(a)\big) \bigg),
$$

where $s_f\colon \pi_0(\mathsf{K}) \to \mathsf{K}$ is given by $s_f(a) = s(f)^{-1} s(fa)$, and $p_k\colon \mathsf{K}_0 \to \mathsf{K}_0/\mathsf{L}$ is the unique continuous homomorphism satisfying $p_k(a)^k = \operatorname{pr}(a)$ for all $a \in \mathsf{K}_0$. It is elementary to show that h is a homomorphism. Hence $\mathsf{E} := \operatorname{pr}^{-1}(h(\pi_0(\mathsf{K})))$ is a finite group intersecting each connected component of K.   q.e.d.

Lemma 7.6.   *Let $\mathsf{H} \subset \mathsf{GL}(l, \mathbb{C})$ be a compact subgroup and suppose that the identity component H_0 is abelian. Then there is a finitely generated dense subgroup $\Phi_2 \subset \mathsf{H}$ and a finitely generated cocompact subgroup Λ_2 such that Λ_2 is invariant under the natural action of Φ_2 on $\mathbb{C}^l$.*

Proof. By Lemma 7.5 there is a finite group $\mathsf{E} \subset \mathsf{H}$ with $\mathsf{H} = \mathsf{E} \cdot \mathsf{H}_0$. Without loss of generality we may assume that H_0 consists of diagonal matrices. By the sublemma there is an algebraic integer $z \in \mathsf{S}^1$ of infinite order. Clearly,

$$
\Phi_2' := \{A \in \mathsf{H}_0 \mid \text{ the Eigenvalues of } A \text{ lie in } \mathbb{Z}[z]\}
$$

is a dense subgroup of H_0. Choose a generator system $e_1, \dots, e_m$ of $\mathbb{C}^l$ which is invariant under the finite group E. Evidently, the finitely generated cocompact subgroup $\Lambda_2 := \operatorname{span}_{\mathbb{Z}}\{Ae_i | A \in \mathsf{H}_0, \ i = 1, \dots, n\}$ is invariant under $\Phi_2 := \mathsf{E} \cdot \Phi_2'$.   q.e.d.

7.6   The proof of Corollary 8

Let G_0 be the identity component of the compact group $\mathsf{G} \subset \mathsf{O}(l)$. Notice that the commutator group $\mathsf{K} := [\mathsf{G}_0, \mathsf{G}_0]$ is a connected compact semisimple Lie group. Choose subgroups $\Phi_1 \subset \mathsf{K}$, $\Lambda_1 \subset \mathbb{C}^l$ and $\mathsf{B} \subset \mathsf{GL}(n, \mathbb{C})$ as in Lemma 7.4. By the very definition of B the group

$$
\mathsf{H} := \mathsf{G} \cap \mathsf{B}
$$

intersects each connected component of G. Using that $\Phi_1 \subset \mathsf{K}$ is dense in K we see that the identity component H_0 of H is contained in the centralizer of K in G. Thus H_0 is the identity component of the center of G_0. In particular, H_0 is a toral subgroup and $\mathsf{G} = \mathsf{H} \cdot \mathsf{K}$.

Choose groups $\Phi_2 \subset \mathsf{H}$ and $\Lambda_2 \subset \mathbb{C}^l$ as stated in Lemma 7.6. The group Λ_2 is finitely generated and Φ_1 leaves the cocompact subgroup Λ_1 invariant. Using Lemma 4.5 we see that

$$\Lambda' := \operatorname{span}_{\mathbb{Z}}\{Av \mid A \in \Phi_1, v \in \Lambda_2\}$$

is finitely generated. By construction Φ_2 normalizes Φ_1, and hence the dense subgroup $\Phi := \Phi_1 \cdot \Phi_2$ of K leaves the cocompact subgroup $\Lambda' \subset \mathbb{C}^l$ invariant.

Recall that $\Phi \subset \mathsf{G} \subset \mathsf{O}(l)$ consists of real matrices. Consequently,

$$\Lambda := \{\operatorname{Real}(v) \mid v \in \Lambda'\}$$

is a finitely generated cocompact subgroup of $\mathbb{R}^l$ which is invariant under Φ. By Corollary 7, Φ is the image of a holonomy representation.

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